Ohms Law

Consider a two species fluid of electrons and singly charged ions. Let the fluid be neutral so that $n_i = n_e$, then the current density is

$$\vec{j} = ne(\vec{u}_i - \vec{u}_e).$$

And the total momentum of the fluid per unit mass is

$$\rho\vec{u} = n(m_i\vec{u}_i + m_e\vec{u}_e).$$

Then it is not hard to solve for the velocities of the two species given the fluids current and momentum.

$$\begin{aligned} \vec{u}_i &= \vec{u} + \frac{m_e\vec{j}}{e\rho}, \\ \vec{u}_e &= \vec{u} - \frac{m_i\vec{j}}{e\rho}. \end{aligned}$$

Now use these expression in Euler's equation, including a frictional force Q, which describes the electron resistivity. This can be modeled by $Q_s = \nu_{s;s'}m_s n_s(\vec{u}_{s'} - \vec{u}_s)$, where $\nu_{s;s'}$ is the frequency of collisions between the two species s and s'. Note that $Q_s + Q_{s'} = 0$, by conservation of momentum (Newton's third Law).

$$\begin{aligned} \partial_t(nm_i\vec{u}_i) + \vec{\nabla}\cdot(\rho_i\vec{u}_i\vec{u}_i) &= -\vec{\nabla}p_i + \frac{ne(\vec{E} + \vec{u}_i\times\vec{B})}{c} + Q_i. \\ \partial_t(nm_e\vec{u}_e) + \vec{\nabla}\cdot(\rho_e\vec{u}_e\vec{u}_e) &= -\vec{\nabla}p_e - \frac{ne(\vec{E} + \vec{u}_e\times\vec{B})}{c} + Q_e. \end{aligned}$$

Now multiple the ion's equation by e/m_i and the electron equation by $-e/m_e$ and summing together we have

$$\partial_t(en(\vec{u}_i - \vec{u}_e)) + \nabla\cdot(en(\vec{u}_i\vec{u}_i - \vec{u}_e\vec{u}_e)) = -\frac{e}{m_i}\vec{\nabla}p_i + \frac{e}{m_e}\vec{\nabla}p_e + \frac{ne^2}{c}\left(\frac{m_i m_e}{m_i + m_e}\vec{E} + \left(\frac{\vec{u}_i}{m_i}\times + \frac{\vec{u}_e}{m_e}\times\right)\vec{B}\right)$$

$$\partial_t\vec{j} + \vec{\nabla}\cdot(\vec{u}\vec{j} + \vec{j}\vec{u}) + \frac{m_e^2 - m_i^2}{e(m_i + m_e)}\vec{\nabla}\cdot(\frac{1}{\rho}\vec{j}\vec{j}) = -\frac{e}{m_i}\vec{\nabla}p_i + \frac{e}{m_e}\vec{\nabla}p_e + \frac{\rho e^2}{m_i m_e c}(\vec{E} + \vec{u}\times\vec{B}) + \frac{ne^2(m_e^2 + m_i^2)}{cm_e m_i}\rho\vec{j}\times\vec{B}.$$

Alfvén waves

Recall the derivation for sound waves in a uniform medium, or at least uniform locally. We shall repeat the procedure including magnetic fields in the medium. If you have never seen the derivation for sound waves, follow along and just set $\vec{B} = 0$ —two derivations for the price of one.

Start with the full suite of equations for ideal magnetohydrodynamics for a adiabatic gas, they are: continuity, Euler's equation, the barotropic equation of state, Ampère's Law without Maxwell extension, Maxwell-Faraday equation, and ideal Ohm's Law.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\vec{j} \times \vec{B}}{\rho c}, \tag{2}$$

$$p = K \, \rho^{\gamma}, \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}, \tag{4}$$

$$\vec{\nabla} \times \vec{E} = \frac{-1}{c} \frac{\partial B}{\partial t}, \tag{5}$$

$$\vec{E} = -\vec{v} \times \vec{B}. \tag{6}$$

We can use then eliminate the electric field, current and pressure to reduce the equations down to

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{7}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\gamma K \vec{\nabla} \rho - \rho \vec{\nabla} \phi + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi c}, \tag{8}$$

$$\vec{\nabla} \times (\vec{v} \times \vec{B}) = \frac{1}{c} \frac{\partial B}{\partial t}. \tag{9}$$

Let's consider initial conditions such that the medium is uniform in ρ, p and $\vec{B}$; then they are perturbed such that

$$\vec{v}(\vec{r}, t) = \delta \vec{v}(\vec{r}, t),$$
$$p(\vec{r}, t) = p_0 + \delta p(\vec{r}, t),$$
$$\vec{B}(\vec{r}, t) = \vec{B}_0 + \delta \vec{B}(\vec{r}, t).$$

Replacing these quantities into the reduced equations lead us to

$$\frac{\partial \delta \rho}{\partial t} + \vec{\nabla} \cdot (\rho_0 \delta \vec{v}) + \mathcal{O}(\delta^2) = 0,$$

$$\rho_0 \frac{\partial \delta v}{\partial t} + \gamma K \vec{\nabla} \delta \rho + \delta \rho \vec{\nabla} \phi - \frac{(\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0}{4\pi c} + \mathcal{O}(\delta^2) = 0.$$

$$\frac{1}{c} \frac{\partial \delta \vec{B}}{\partial t} = \vec{\nabla} \times (\delta v \times \vec{B}_0) + \mathcal{O}(\delta^2).$$

Now we will take these equations to linear order int δ. We will then use the equations to eliminate $\delta \vec{B}$ and $\delta \rho$, by taking the gradient of the linearized continuity equation and multiplying it by $-\gamma K$, and adding it to the time derivative of the linearized Euler's equation.

$$\rho_0 \frac{\partial^2 \delta \vec{v}}{\partial t^2} - \gamma K \vec{\nabla}(\vec{\nabla} \cdot (\rho_0 \delta \vec{v})) - \frac{(\vec{\nabla} \times \frac{1}{c}\frac{\partial \delta \vec{B}}{\partial t}) \times \vec{B}_0)}{4\pi} = 0.$$

Notice, that time indepdent potentials drop out. Now substituting in our linearized Ampère's equation into this expression, and dividing through by ρ, we arrive at

$$\partial_t^2 \delta \vec{v} - \gamma K \vec{\nabla}(\vec{\nabla} \cdot \delta \vec{v}) - \frac{B_0^2}{4\pi\rho} \left(\vec{\nabla} \times \left(\vec{\nabla} \times (\delta \vec{v} \times \hat{e}_B)\right)\right) \times \hat{e}_B = 0.$$

You may already know that $\gamma K = c_s^2$, where c_s is the speed of sound, so we are tempted to view the coefficient on the second term as some velocity too, which I will label $c_A^2 = \frac{B_0^2}{4\pi\rho}$. Now to understand what these "velocities" are we will solve for a dispersion relation from the above the equation by taking $\delta \vec{v}(\vec{r}, t) = \vec{v} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.

$$-\omega^2 \delta \vec{v} + c_s^2 \vec{k}(\vec{k} \cdot \delta \vec{v}) + c_A^2 \left(\vec{k} \times \left(\vec{k} \times (\delta \vec{v} \times \hat{e}_B)\right)\right) \times \hat{e}_B = 0.$$

Expanding the quadruple cross product

$$\begin{aligned}
\hat{e}_B \times \left(\vec{k} \times \left(\vec{k} \times (\delta \vec{v} \times \hat{e}_B)\right)\right) &= \epsilon_{ijk}(\hat{e}_B)_j \left(\epsilon_{lmj}(\vec{k})_l \left(\epsilon_{nom}(\vec{k})_n \left(\epsilon_{pqo}(\delta\vec{v})_p(\hat{e}_B)_q\right)\right)\right) \\
&= \left(\epsilon_{ikj}\epsilon_{lmj}(\hat{e}_B)_j(\vec{k})_l\right)\left(\epsilon_{nmo}\epsilon_{pqo}(\vec{k})_n(\delta\vec{v})_p(\hat{e}_B)_q\right) \\
&= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})(\hat{e}_B)_j(\vec{k})_l(\delta_{np}\delta_{mq} - \delta_{nq}\delta_{mp})(\vec{k})_n(\delta\vec{v})_p(\hat{e}_B)_q \\
&= \left((\hat{e}_B)_l(\vec{k})_l\delta_{km} - (\hat{e}_B)_m(\vec{k})_k\right)\left((\vec{k})_n(\delta\vec{v})_n(\hat{e}_B)_m - (\vec{k})_n(\delta\vec{v})_m(\hat{e}_B)_n\right) \\
&= (\hat{e}_B \cdot \vec{k})(\vec{k} \cdot \delta\vec{v})\hat{e}_B - (\hat{e}_B \cdot \vec{k})(\vec{k} \cdot \hat{e}_B)\delta\vec{v} \\
&\quad -(\hat{e}_B \cdot \hat{e}_B)(\vec{k} \cdot \delta\vec{v})\vec{k} + (\hat{e}_B \cdot \delta\vec{v})(\vec{k} \cdot \hat{e}_B)\vec{k}.
\end{aligned}$$

Thus we have arrived at the dispersion equation for Alfvèn waves,

$$-\omega^2 \delta \vec{v} + (c_s^2 + c_A^2)\vec{k}(\vec{k} \cdot \delta \vec{v}) + c_A^2(\hat{e}_B \cdot \vec{k})\left((\vec{k} \cdot \hat{e}_B)\delta\vec{v} - (\vec{k} \cdot \delta\vec{v})\hat{e}_B - (\hat{e}_B \cdot \delta\vec{v})\vec{k}\right) = 0. \tag{10}$$

Now to simply the math, chose $\vec{B}$ to be along the z–axis, and rotate the x–y plane such that $\vec{k}$ lies in the x–z plane. Thus

$$\begin{aligned}
\vec{B} &= B_0 \hat{e}_z, \\
\vec{k} &= k\,(\sin(\theta)\hat{e}_x + \cos(\theta)\hat{e}_z), \\
\delta\vec{v} &= \delta v_x \hat{e}_x + \delta v_y \hat{e}_y + \delta v_z \hat{e}_z.
\end{aligned}$$

Then,

$$\begin{aligned}
(\hat{e}_B \cdot \vec{k}) &= k\cos\theta, \\
(\vec{k} \cdot \delta\vec{v}) &= k\,(\sin(\theta)\delta v_x + \cos(\theta)v_z), \\
(\hat{e}_B \cdot \delta\vec{v}) &= \delta v_z.
\end{aligned}$$

$$-\omega^2\delta\vec{v} + k(c_s^2 + c_A^2)\left(\sin(\theta)\delta v_x + \cos(\theta)v_z\right)\vec{k} + c_A^2 k\cos(\theta)\left(k\cos\theta\delta\vec{v} - k\left(\sin(\theta)\delta v_x + \cos(\theta)v_z\right)\hat{e}_B - \delta v_z\vec{k}\right) = 0.$$

$$\left(-\omega^2 + k^2 c_A^2 \cos^2\theta\right)\delta\vec{v} + k\left((c_s^2 + c_A^2)\delta v_x \sin(\theta) + c_s^2 \delta v_z \cos(\theta)\right)\vec{k} - c_A^2 k^2 \cos(\theta)\left(\sin(\theta)\delta v_x + \cos(\theta)\delta v_z\right)\hat{e}_B = 0. \tag{11}$$

This now allows us to look at each component of $\delta\vec{v}$ and written in matrix notation

$$M = \begin{bmatrix} -\omega^2 + k^2c_s^2\sin^2\theta + k^2c_A^2 & 0 & k^2c_s^2\cos(\theta)\sin(\theta) \\ 0 & -\omega^2 + k^2c_A^2\cos^2(\theta) & 0 \\ k^2c_s^2\sin(\theta)\cos(\theta) & 0 & -\omega^2 + k^2c_s^2\cos^2(\theta) \end{bmatrix} \begin{bmatrix} \delta v_x \\ \delta v_y \\ \delta v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the characteristic equation for ω^2 we will find three solutions.

$$\det(M) = \left(-\omega^2 + k^2c_A^2\cos^2(\theta)\right)\left(\omega^4 - k^2(c_s^2 + c_A^2)\omega^2 + k^4c_A^2c_s^2\cos^2(\theta)\sin^2(\theta)\right) = 0.$$

$$\begin{aligned} \omega^2 &= k^2c_A^2\cos^2(\theta), \\ \omega^2 &= \tfrac{1}{2}k^2\left((c_s^2 + c_A^2) \pm \sqrt{(c_s^2 + c_A^2)^2 - 4c_s^2c_A^2\cos^2(\theta)}\right). \end{aligned}$$

These are in general the dispersion relations for magnetosonic disturbances in a magnetic fluid. The first one is known as the Alfvèn mode, and the second two are called the fast (+) and slow (−) modes. For a general disturbance all three modes are present. However, it is physically enlightening to examine two specific cases; when the $\vec{k}$ is parallel or perpendicular to the magnetic field.

When the $\vec{k}$ is parallel to the magnetic field, $\cos(\theta) = 1$. Thus we have

$$\begin{aligned} \omega^2 &= k^2c_A^2, \\ \omega^2 &= \tfrac{1}{2}k^2\left((c_s^2 + c_A^2) \pm (c_s^2 - c_A^2)\right). \end{aligned}$$

Thus we see that there is really only two modes with dispersion relations $\omega^2 = k^2c_s^2$ and $\omega^2 = k^2c_A^2$. Thus we can see right away that there are two types of disturbances with velocities c_s and c_A, respectively. Moreover, looking back at equation (**??**) and substituting in $\theta = \pi$, and noting that $\vec{k} = k\hat{e}_B$, we find that

$$(-\omega^2 + k^2c_A^2)\delta\vec{v} + (c_s^2 - c_A^2)(\vec{k}\cdot\delta\vec{v})\vec{k} = 0.$$

Consider the component of the disturbance propagation parallel to $\vec{k}$, then this reduces to $(-\omega^2 + c_s^2k^2)\delta v_k = 0$, thus this wave is longitudinal wave that travels at the speed of sound, as expected. The component perpendicular to $\vec{k}$ has $(-\omega^2 + c_A^2k^2)\delta v_k = 0$. This is a transverse wave that propagates at the Alfvèn speed; thus we call this the Alfvèn velocity,

$$c_A = \frac{B_0}{\sqrt{4\pi\rho}}. \tag{12}$$

Now, looking at the linearized Ampère's equation, we find that if there is a magnetic field disturbance that it comes from the perpendicular component of the velocity disturbance. Thus the Alfvèn waves do not give rise to a magnetic field disturbance.

For a completely perpendicular velocity disturbance to the magnetic field vector parallel to the magnetic field is said to be in the Alfvèn mode. From the dispersion relationship the phase speed is $\pm c_A|\cos(\theta)|$ and a group speed of $\pm c_A$, as already noted from the wave relation.

For the perpendicular propagating disturbance we the only non-zero mode is $\omega^2 = k^2(c_s^2 + c_A^2)$. This could have easily been seen from equation (**??**), given $\hat{e}_B \cdot \vec{k} = 0$. Thus we have a wave that a wave that propagates with speed $\sqrt{c_s^2 + c_A^2}$.