Restricted three body problem

We begin by introducing the restricted three body problem. In this scenario we have two massive objects in circular orbits about their center of mass, with a third non-gravitationally interacting body. Our task in this section is to write out the equation of motion for the third body. Our analysis will be carried out in the non-internal frame co-rotating with the two massive objects, such that they are at rest along the x–axis, and the axis of rotation is align with the z–axis.

In non-intertial frames, non-intertial forces arise; specifically for a frame rotating with angular frequency Ω there are the centrifugal, Coriolis and Euler forces. From a simple analysis of a how a time derivative transforms between an inertial and rotating frame, one could discover that

$$\left(\frac{\partial}{\partial t}\right)_I \to \left(\frac{\partial}{\partial t}\right)_R + \vec{\Omega} \times \quad .$$

Where the subscripts "I" and "R" are in turn for inertial and rotating. Thus, the kinematics are changed in a rotating frame by the additional $\vec{\Omega}_0 \times \_$ term, giving rise to the three previously mention non-inertial forces when writing out Newton's Second Law. These are, per unit mass

$$\left(\frac{\partial^2 \vec{x}}{\partial t^2}\right)_I = \left(\frac{\partial^2 \vec{x}}{\partial t^2}\right)_R + \vec{\Omega} \times \vec{\Omega} \times \vec{x} + 2\vec{\Omega} \times \left(\frac{\partial \vec{x}}{\partial t}\right)_R + \frac{\partial \vec{\Omega}}{\partial t} \times \vec{x}.$$

Thus in a rotating frame we have the forces appearing as

$$\left(\frac{\partial^2 \vec{x}}{\partial t^2}\right)_R = \left(\frac{\partial^2 \vec{x}}{\partial t^2}\right)_I - \vec{\Omega}_0 \times \vec{\Omega}_0 \times \vec{x} - 2\vec{\Omega}_0 \times \left(\frac{\partial \vec{x}}{\partial t}\right)_R - \frac{\partial \vec{\Omega}_0}{\partial t} \times \vec{x}.$$

On the righthand side, moving left to right these are the: inertial forces (all forces present in the inertial frame, $-\nabla\phi$), the centrifugal force, Coriolis force and Euler force. Back to the three body problem as described our frame has $\vec{\Omega} = \{0, 0, \Omega_0\}$, thus there is no Euler force ($\partial_t \vec{\Omega} = 0$), and both the Coriolis and centrifugal forces are only present in the x and y components of motion ($\vec{\Omega} \times \hat{z} = 0$). The next step of writing this out in component can be done from the previous equation and recalling that $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{z} \times \hat{y} = -\hat{x}$. Thus

$$\begin{aligned}
\ddot{x} &= -\vec{\nabla}\phi \cdot \hat{x} + \Omega_0^2 x + 2\Omega_0 \dot{y}. \\
\ddot{y} &= -\vec{\nabla}\phi \cdot \hat{y} + \Omega_0^2 y - 2\Omega_0 \dot{x}. \\
\ddot{z} &= -\vec{\nabla}\phi \cdot \hat{z}.
\end{aligned}$$

So far we have not really thought about the other two bodies except for they interact with each other and are fixed along the x–axis in the co-rotating frame. However, these two bodies also impose forces on our third body, but the third body imposes no appreciable force on them. This is built into the $-\vec{\nabla}\phi$ term, which we will now write for gravitational forces. Recall the graviton potential energy and write the distance to the two bodies as $r_1 = \sqrt{(x + \mu_2 a)^2 + y^2 + z^2}$, where $\mu_2 = \frac{m_2}{m_1+m_2}$, and similarly $r_2 = \sqrt{(x - \mu_1 a)^2 + y^2 + z^2}$, where $\mu_1 = \frac{m_1}{m_1+m_2}$.

$$
\begin{aligned}
-\vec{\nabla}\phi &= -(\partial_x, \partial_y, \partial_z)\left[\frac{-GM_1}{r_1} + \frac{-GM_2}{r_2}\right] \\
&= \left(\left[\frac{-GM_1(x+\mu_2 a)}{r_1^3} + \frac{-GM_2(x-\mu_1 a)}{r_2^3}\right], \left[\frac{-GM_1}{r_1^3} + \frac{-GM_2}{r_2^3}\right]y, \left[\frac{-GM_1}{r_1^3} + \frac{-GM_2}{r_2^3}\right]z\right)
\end{aligned}
$$

Combining all of this together we have the equation of motion for the third body in the rotating reference frame, with origin center on the barycenter of the two massive bodies is as follows

$$
\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} - \Omega_0^2 x &= -\left[\frac{GM_1(x+\mu_2 a)}{r_1^3} + \frac{GM_2(x-\mu_1 a)}{r_2^3}\right], \\
\ddot{y} + 2\Omega_0\dot{x} - \Omega_0^2 y &= -\left[\frac{GM_1}{r_1^3} + \frac{GM_2}{r_2^3}\right]y, \\
\ddot{z} &= -\left[\frac{GM_1}{r_1^3} + \frac{GM_2}{r_2^3}\right]z.
\end{aligned}
$$

Hill Equations: Local to Second Body Approximation

Consider the case where the two massive bodies are a star and planet, with the star much more massive than the planet. What does this imply for the third body? Far away from the planet the third body will not feel an appreciable force from the planet and will effectively just be a small perturbation from a Keplerian orbit around the massive star. However, we will take a look at what happens when the third body approaches the planet. Let's call $\mu_1 = 1-\epsilon$ and $\mu_2 = \epsilon$, where ϵ is presumed to be a small number when the star is much more massive than the planet.

We will shift our coordinate system such that is aligns with the planet's core, i.e., $x \to x + a(1-\epsilon)$, additionally we will consider distances from the second body much small than the distance between the two massive bodies, a, i.e., $x << a$ (Note this x and from here on out is the shifted x centered on the planet).

$$
\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} - \Omega_0^2(x + a(1-\epsilon)) &= -\left[\frac{GM_1(x+a)}{\sqrt{(x+a)^2+y^2+z^2}^3} + \frac{GM_2(x)}{\sqrt{x^2+y^2+z^2}^3}\right], \\
\ddot{y} + 2\Omega_0\dot{x} - \Omega_0^2 y &= -\left[\frac{GM_1}{\sqrt{(x+a)^2+y^2+z^2}^3} + \frac{GM_2}{\sqrt{x^2+y^2+z^2}^3}\right]y, \\
\ddot{z} &= -\left[\frac{GM_1}{\sqrt{(x+a)^2+y^2+z^2}^3} + \frac{GM_2}{\sqrt{x^2+y^2+z^2}^3}\right]z.
\end{aligned}
$$

Let's focus on the centrifugal term and the stellar gravitational force for the x equation of motion, bringing both on the right hand side. Expand for small x, y, and z.

$$
\begin{aligned}
& \Omega_0^2(x+a(1-\epsilon)) - \frac{GM_1(x+a)}{\sqrt{(x+a)^2+y^2+z^2}^3} \\
= & \quad \Omega_0^2(x+a(1-\epsilon)) - \frac{GM_1(x+a)}{a^3}\frac{1}{\sqrt{(1+\frac{x}{a})^2+y^2+z^2}^3}, \\
= & \quad \Omega_0^2(x+a(1-\epsilon)) - \frac{GM_1(x+a)}{a^3}\left[1-3\frac{x}{a}+\mathcal{O}(x^2)+\mathcal{O}(y^2)+\mathcal{O}(z^2)\right],
\end{aligned}
$$

From Kepler's Second Law,

$$\frac{GM_1}{a^3} = \frac{GM}{a^3}(1-\epsilon) = \Omega_0^2(1-\epsilon).$$

$$
\begin{aligned}
& \Omega_0^2(x+a(1-\epsilon)) - \frac{GM_1(x+a)}{a^3}\left[1-3\frac{x}{a}+\mathcal{O}(x^2)+\mathcal{O}(y^2)+\mathcal{O}(z^2)\right], \\
= & \quad \Omega_0^2(x+a(1-\epsilon)) - \Omega_0^2(1-\epsilon)(x+a)\left[1-3\frac{x}{a}+\mathcal{O}(x^2)+\mathcal{O}(y^2)+\mathcal{O}(z^2)\right], \\
= & \quad \Omega_0^2\left[x+a-a\epsilon-x+\epsilon x-a+a\epsilon+3x-3x\epsilon\right]+\mathcal{O}(x^2)+\mathcal{O}(y^2)+\mathcal{O}(z^2), \\
= & \quad \Omega_0^2\left[3x-2x\epsilon\right]+\mathcal{O}(x^2)+\mathcal{O}(y^2)+\mathcal{O}(z^2).
\end{aligned}
$$

If we take ϵ to be $\mathcal{O}(\frac{x}{a})$, then to first order the tidal term (centrifugal and stellar forces) is $3\Omega_0^2 x$.

$$
\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} &= \left[3\Omega_0^2 - \frac{GM_2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right]x, \\
\ddot{y} + 2\Omega_0\dot{x} &= -\frac{GM_2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\,y, \\
\ddot{z} &= -\left[\frac{GM_1}{\left(\sqrt{(x+a)^2+y^2+z^2}\right)^3}+\frac{GM_2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right]z.
\end{aligned}
$$

Generalizing for non-Keplerian systems

We have assumed a Keplerian rotation curve, which follows naturally from a singular massive gravitating body; however, this might not always be the case so let's take a step back and generalize our equations. We still wish to have circular orbits, such that in a frame rotating at Ω_0, a position r_0 is fixed. Thus we will need a potential that satisfies

$$-\vec{\nabla}\phi\Big|_{r_0} = -\Omega_0^2\vec{r}_0.$$

The equation of motion in this case are

$$\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} &= (\Omega_0^2 - \Omega^2)(x + r_0). \\
\ddot{y} + 2\Omega_0\dot{x} &= (\Omega_0^2 - \Omega^2)y. \\
\ddot{z} &= -\Omega^2 z.
\end{aligned}$$

Let's define a parameter q as

$$q \equiv -\frac{\partial \ln \Omega}{\partial \ln r}.$$

Examples of the parameter are: solid body rotation ($q = 0$), flat rotation curves ($q = 1$), uniform angular momentum disc ($q = 2$), and Keperlian disc ($q = 3/2$). Angular rotation profiles that follow this definition are of the form

$$\Omega(r) = \Omega_0 \left(\frac{r_0}{r}\right)^q.$$

Shifting from the origin to our region of interest, the shearing box, we can rewrite this as

$$\Omega(x, y, z) = \Omega_0 \left(\frac{r_0}{\sqrt{(x + r_0)^2 + y^2 + z^2}}\right)^q.$$

Since we are considering a local region with linear length much less than r_0, we will Taylor expand Ω around this region to first order.

$$\begin{aligned}
\Omega(\Delta x, \Delta y, \Delta z) &= \Omega(0,0,0) + \Delta x \frac{\partial \Omega}{\partial x}\Big|_{r_0} + \Delta y \frac{\partial \Omega}{\partial y}\Big|_{r_0} + \Delta z \frac{\partial \Omega}{\partial z}\Big|_{r_0} + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2) + \mathcal{O}(\Delta z^2), \\
&= \Omega_0 \text{ - } \frac{q\,\Omega_0}{r_0}\Delta x + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2) + \mathcal{O}(\Delta z^2).
\end{aligned}$$

Squaring this, keeping only linear terms, and putting it into our equations of motion yield

$$\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} &= 2q\Omega_0^2 x. \\
\ddot{y} + 2\Omega_0\dot{x} &= 0. \\
\ddot{z} &= -\Omega_0^2 z.
\end{aligned}$$

$$\frac{\partial^2 \vec{x}}{\partial t^2} = -2\vec{\Omega}_0 \times \vec{v} + \Omega_0^2(2qx\hat{x} - z\hat{z}).$$

Shearing Box

The past two decades have seen recent application of this approximation in MHD disc simulations, dubbed the shearing box approximation. We will now consider the implication of this approximation in continuum mechanics. Let's start by review the momentum equation. Widely stated as

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \vec{f}.$$

However, let's rewrite this equation in conservative form. We will start by noting this is the momentum equation, so we will want to have the momentum being differentiated with respect to time. From the derivation of this form we used the continuum equation to disassemble that term, so now let's reassemble it. First multiple the equation by ρ.

$$\rho\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})(\rho\vec{u}) = -\vec{\nabla}p + \rho\vec{f}.$$

Now recall the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0.$$

Multiplying this by $\vec{u}$ and adding it to our momentum equation nets us

$$\rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\frac{\partial\rho}{\partial t} + \vec{u}\left(\vec{\nabla}\cdot(\rho\vec{u})\right) + (\vec{u}\cdot\vec{\nabla})(\rho\vec{u}) = -\vec{\nabla}p + \rho\vec{f}.$$

Recall the divergence of a tensor, using Einstein summation notation,

$$\vec{\nabla}\cdot\overline{\overline{\tau}} = \frac{\partial\tau_{ij}}{\partial x_j}\,\mathbf{e}_i.$$

If $\overline{\overline{\tau}}$ is a dyadic tensor ($\tau_{ij} = a_i b_j$), then

$$\vec{\nabla}\cdot\overline{\overline{\tau}} = (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}(\vec{b}\cdot\vec{\nabla}).$$

Note that this appears in our momentum equation and thus we can simply the equation to compactly write

$$\frac{\partial\rho\vec{u}}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = -\vec{\nabla}p + \rho\vec{f}.$$

This is the conservative form which has the flux of momentum ($\rho\vec{u}\vec{u}$), and the sources terms, both internal ($-\vec{\nabla}p$) and external ($\rho\vec{f}$). Replacing f with our local approximation forces we found for a general system with shear parameter q

$$\frac{\partial\rho\vec{u}}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}\vec{u} + p\overline{\overline{I}}) = -2\rho\vec{\Omega}_0\times\vec{v} + \rho\Omega_0^2(2qx\hat{x} - z\hat{z}).$$

Slight modification to the energy equation we simply need to recall that $\frac{\partial E}{\partial t} = \vec{f}\cdot\vec{u}$ and we get

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E\vec{u} + p\overline{\overline{I}}\cdot\vec{u}) = \Omega_0^2\rho\vec{u}\cdot(2qx\hat{x} - z\hat{z}).$$

Notice the Coriolis force does no work, thus does not alter the energy equation.