Rankie-Hugoniot relations

The equations that govern fluids are capable of discussing discontinuities in the field. Let's take a look at interfaces between two fields and see how the system behaves. The relations derived are known as the Rankie-Hugoniot relations.

See figure below for the setup of the two fields in the problem. We will consider the fluid parameters ρ, p and the flow velocity u. Note for simplicity we carry out the work in the shock frame, and can boost to other frames later as desired.

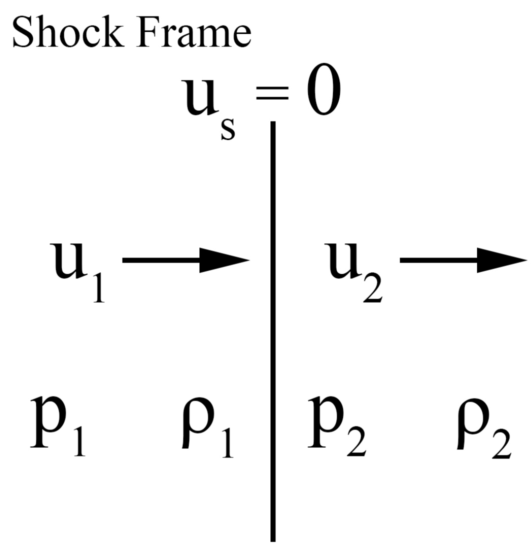

Figure 1: Field 1 on the left and field 2 on the right. The black bar between the two is the interface, possibly a shock front.

Consider the one dimensional continuity equation, which we will justify using briefly,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho\, u_x\right) = 0, \tag{1}$$

at an interfaces of two uniform fields: field 1 and field 2. The properties of these fields are subscripted such that ρ_1 is the density in field 1, $u_{x,2}$ is the x velocity of the fluid in field 2, etc. Now integrate the continuity equation around a layer of thickness $d\ell$ about the interface of the two fields.

$$\int_{-d\ell/2}^{d\ell/2} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho\, u_x\right) \mathrm{d}x = 0.$$

$$\int_{0}^{-d\ell/2} \frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left(\rho_1\, u_{x,1}\right) \mathrm{d}x = \int_{0}^{d\ell/2} \frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}\left(\rho_2\, u_{x,2}\right) \mathrm{d}x.$$

Now the time derivative term is zero since as $d\ell \to 0$ so does the value of the integral and the value of that does not change much over time, therefore we are okay to ignore it. The remaining term is a simply application of the fundamental theorem of calculus, resulting in

$$\rho_1\, u_{x,1} = \rho_2\, u_{x,2}. \tag{2}$$

Where these values are evaluated $d\ell/2$ distance away from the interface in each field. Thus as $d\ell \to 0$ these are the left and right limiting values of the interface, possibly a discontinuity. This is the first Rankie-Hugoniot relation.

Next we do the same for the momentum equation, written in component form with a continuous potential

$$\frac{\partial \rho \vec{u}_i}{\partial t} + \partial_j \left(\rho u_j u_i + p\delta_{ij}\right) = -\rho \partial_i \Psi. \tag{3}$$

Since the shock plane is the surface of interest in the problem, let's only consider momentum which is transferred through the shock surface, limiting us to $j = x$. Now consider the x component of that momentum transport, that is $i = x$

$$\frac{\partial \rho u_x}{\partial t} + \frac{\partial}{\partial x}\left(\rho\, u_x\, u_x + p\right) = -\rho \frac{\partial \Psi}{\partial x}. \tag{4}$$

We can see the maths will be the same, except we will give special attention to the continuous potential. We will start by integrating the RHS of the previous equation by parts

$$\int_1^2 \rho \frac{\partial \Psi}{\partial x} \mathrm{d}x = \rho\Psi\Big|_1^2 - \int_1^2 \frac{\partial \rho}{\partial x} \Psi \mathrm{d}x. \tag{5}$$

Since Ψ is continuous and as $d\ell \to 0$, then the value of Ψ is approximately constant across $d\ell$, thus to good approximation we can pull Ψ out of the integral, it is equal to the approximation that $\Psi_1 = \Psi_2$ and will simply the first term too. Note the second term on the LHS is then just another application of the fundamental theorem of calculus

$$\int_1^2 \rho \frac{\partial \Psi}{\partial x} \mathrm{d}x = \Psi(\rho_2 - \rho_1) - \Psi(\rho_2 - \rho_1) = 0. \tag{6}$$

We have just shown that any continuous potential maybe acting on our system and will have no effect on the second Rankie-Hugoniot relations. Going back to equation (4) we can apply all of our knowledge to see

$$\rho_1\, u_{x,1}^2 + p_1 = \rho_2\, u_{x,2}^2 + p_2. \tag{7}$$

This is the second Rankie-Hugoniot relation.

As a side note if we consider the off diagonal terms from the momentum equation ($i = y$ or $i = z$), terms were the stress tensor does not include the thermal pressure, we see that the components of the Reynolds stress tensor are conserved across the interface ($\rho\, u_j\, u_i$). However from the continuity equation we know that $\rho\, u_x$ is conserved, thus we can see that u_y is conserved and so is u_z. It is convent to set these to zero, at no loss of generality, when working out the physics of a non-relativistic shock.

One could also use the symmetry of an infinite plane to argue these values are constant. Assume u_y changed across the interface, then by the symmetry of the plane, every streamline across the interface would have to have the same Δu_y across the interface. Now rotate our coordinate frame about the normal of the interface by a degree θ. As we rotate our frame we will need to require $\Delta u_y = \Delta u_{y'}$, since by the symmetry of the plane there is no distinction between the two frames. To keep this $\Delta u_{y'} = \Delta u_y$ as is required by our assumption what does Δu_z need to be? A simple rotation has $\Delta u_{y'} = \Delta u_y \sin(\theta) + \Delta u_z \cos(\theta)$, therefore $\Delta u_z = \Delta u_y(1 - \cos(\theta))/\sin(\theta)$, which is not a constant unless $\Delta u_y = 0$, which violates our original assumption. Therefore $\Delta u_y = 0$ and $\Delta u_z = 0$.

Lastly we deal with the energy equation

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+p)u] + \rho \dot{Q}_{cool} - \rho \frac{\partial \Psi}{\partial t} = 0 \tag{8}$$

We will simply the energy equation by saying we are working with an adiabatic gas, and that the potentials are time independent. For an Adiabatic gas $\dot{Q}_{cool} = 0$, and for time independent potentials $\partial_t \Psi = 0$. Using the same math rigmarole we get that the conserved value across the interface is

$$(E_1 + p_1)u_1 = (E_2 + p_2)u_2. \tag{9}$$

Now lets say the energy of the gas is simply kinetic, $1/2\rho u^2$, internal, $\rho\epsilon$, and energy from continuous, time independent potentials, $\rho\Psi$,

$$E = \left[\frac{1}{2}u^2 + \epsilon + \Psi\right]\rho. \tag{10}$$

The continuous potentials cancel on both sides since the interface is infinitesimally small and they have the same values immediately on either side of the interface. Then

$$(\frac{1}{2}u_1^2 + \epsilon_1 + \frac{p_1}{\rho_1})\rho_1\, u_1 = (\frac{1}{2}u_2^2 + \epsilon_2 + \frac{p_2}{\rho_2})\rho_2\, u_2. \tag{11}$$

Now from our first Rankie-Hugoniot relation, we see the ρu terms on both side cancel. Next we will also rewrite the internal energy as $\epsilon = \frac{1}{\gamma-1}\frac{p}{\rho}$, a completely general property for any gas. Note $1 + \frac{1}{\gamma-1} = \frac{\gamma}{\gamma-1}$. Lastly we will find it interesting to recast the equation in terms of speed of sound, for adiabatic gas $c_s^2 = \gamma\frac{p}{\rho}$.

$$\frac{1}{2}u_{x,1}^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_{x,2}^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}, \tag{12}$$

$$\frac{1}{2}u_{x,1}^2 + \frac{c_{s,1}^2}{\gamma-1} = \frac{1}{2}u_{x,2}^2 + \frac{c_{s,2}^2}{\gamma-1}.$$

This will be our third and final Rankie-Hugoniot relations, which all together are (2), (7) and (12).

Jump Conditions

From the Rankie-Hugoniot relations we can get some interesting physical results from some algebra manipulation. In this derivation the symmetry of the equations is preserved, until the final step when we wish to break it for the desired result.

Let's relabel the first relation as $j = \rho_1\, u_1 = \rho_2\, u_2$, a typical looking density current. Now rewriting the second relation in suit results in

$$p_1 + \frac{j^2}{\rho_1} = p_2 + \frac{j^2}{\rho_2}.$$

Solve for j^2

$$j^2 = \frac{p_2 - p_1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}}.$$

The third relationship rewritten in terms of j gives

$$\frac{1}{2}\frac{j^2}{\rho_1^2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}\frac{j^2}{\rho_2^2} + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}.$$

Now from this equation we will derive the jump conditions, making the appropriate substitution of j^2 and doing some algebra.

$$\frac{j^2}{2}\left[\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right] = \frac{\gamma}{\gamma - 1}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right],$$

$$\frac{1}{2}\frac{p_2 - p_1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}}\left[\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right] = \frac{\gamma}{\gamma - 1}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right],$$

Note: $\frac{x^2-y^2}{x-y} = \frac{(x+y)(x-y)}{x-y} = (x + y)$,

$$\frac{1}{2}(p_2 - p_1)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] = \frac{\gamma}{\gamma - 1}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right],$$

Now group density on each side,

$$\frac{1}{\rho_1}\left[\frac{1}{2}(p_2 - p_1) + \frac{\gamma}{\gamma - 1}p_1\right] = \frac{1}{\rho_2}\left[\frac{1}{2}(p_1 - p_2) + \frac{\gamma}{\gamma - 1}p_2\right],$$

Note: $\frac{1}{2} + \frac{\gamma}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}$,

$$\frac{1}{\rho_1}\left[\left(\frac{\gamma + 1}{\gamma - 1}\right)p_1 + p_2\right] = \frac{1}{\rho_2}\left[\left(\frac{\gamma + 1}{\gamma - 1}\right)p_2 + p_1\right].$$

Finally arriving at,

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma + 1)p_1 + (\gamma - 1)p_2} = \frac{u_1}{u_2}. \quad (13)$$

These are known as the jump conditions.

Applications

Shocks happen whenever two fields interface with each and create a discontinuity. You might recall that shocks are commonly caused by SN in astronomy, however distinguish that a SN itself does not generate a shock. If a SN went off in a vacuum we would not observe a shock! or those pretty super bubbles. What causes the shock is the SN slamming into the ISM, which is another field with different parameters and at their interface we have a discontinuity. A blast wave is a shock such that the pressure inside the shock front is immensely greater than the ambient medium. In this case we have the condition $p_2 >> p_1$***, this is called a strong shock.***

Consider a strong shock for a monatomic gas ($\gamma = \frac{5}{3}$). From the jump conditions the limit of a strong shock is that

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} = \frac{u_1}{u_2}. \tag{14}$$

For $\gamma = \frac{5}{3}$, we get that this ratio is 4.

Note that all this work has been done with respect to the shocks reference frame. That is, that the shock is stationary and the gas has been passing through. However, typically we are not in the shocks reference frame when we are observing (unless we are riding on the blast wave). Thus is it useful be able to frame shift into the "lab" frame.

At this point it is best to draw two frames of references, a lab frame where the ambient medium is typically taken to be zero, and the shock frame, where the shock is stationary. In the shock frame, we know that the upstream fluid is moving four times faster than the downstream fluid, in the case of a strong shock.

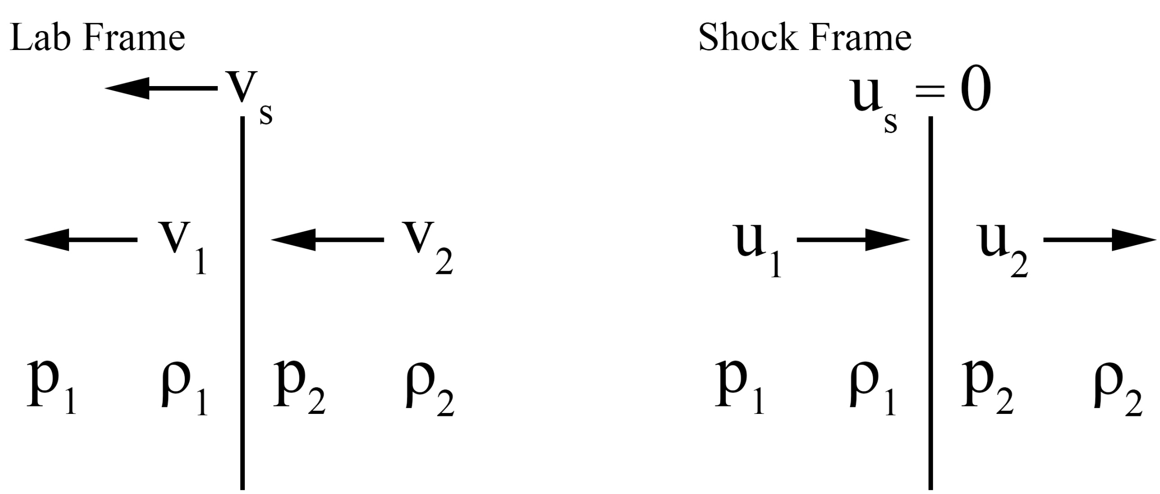

Figure 2: Left: The lab frame of a shock. Right: Shock frame of the shock.

In the lab frame, given the the ambient medium is stationary, the shock front is moving at the speed of the upstream velocity in the shocks frame. The velocity of the shocked gas is also the velocity of the shock minus the downstream velocity. This all translates to

Lab:

$$v_s = -u_1,$$

$$v_g = u_2 + v_s,$$

Shock:

$$u_1 = 4u_2.$$

Where v_s is the shock's speed, u_1 is the upstream velocity, u_2 is the downstream speed. Putting this all together we get that the shocked gas is moving at $\frac{3}{4}v_s$.

Conditions for Shocks

We have seen the relations which govern how shocked gas behaves, specifically it's properties are set by the field is is crashing into, not the field which is "generating" the shock. But what determines when a shock happens and when it doesn't?