Wind Power(ed)

For a shock to be powered by wind, the condition that the wind is in contact with the shock needs to be meet. If the shock is moving faster than the wind, then the wind is not dumping momentum into the shock, and therefore the shock is not powered by a wind (although maybe it was generated by one). Thus we will work out the condition that the shock is powered by the wind.

To find out the condition that the wind be in contact with the shock we will use the jump conditions and the 2nd Rankie-Hugoniot relation (See 'Overview of Hydrodynamic Shocks' for details). We use the convention that $p_2 \geq p_1$. We will also work under the assumption that the properties of the ambient (subscripts denoted with 1) and the wind are known (subscripts denoted with 'w'). We will take the properties of the shocked gas (subscripts denoted with 2) as unknowns. These relations apply in the shock frame and respectively are

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2} \equiv X, \tag{1}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2. \tag{2}$$

Note that for $\gamma = \frac{5}{3}$, that $1 \leq X \leq 4$. Recall that ρ, p and γ are all Galilean invariants. To get into this frame from the lab frame we subtract the speed at which the shock is propagating, which I will call the shock velocity from here on out. Typically we know the lab velocities so we will need to rewrite the shock frame's velocity in terms of these. They are

$$u_1 = v_1 - v_s,$$

$$u_2 = v_2 - v_s.$$

Note I have adapted the convention that the lab frame the velocities are denoted by v, while in the shock frame they are denoted by u.

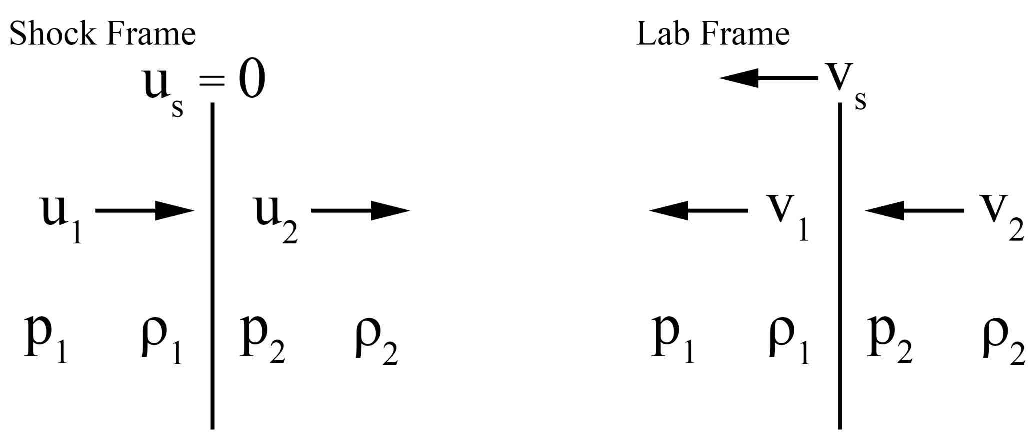

Figure 1: Left is the schematics for the shock frame, and the right for the lab frame. The arrows denote the sign convention. In this case the shock is faster than both the shocked gas and ambient, which is physically sound.

Now the condition that the shock is driven by the a wind requires that $v_w \geq v_2$, thus from our relations we will want to solve for v_2. However, the math works out the same if we solve for v_s, which itself a something we might wish to known, and use the relations to relate it to v_2.

Being by replacing the shocked density and velocity with its relation to the ambient density and velocity using (1).

$$\rho_1 u_1^2 + p_1 = X\rho_1 \left(\frac{1}{X}u_1\right)^2 + p_2,$$

$$\left(1 - \frac{1}{X}\right)\rho_1 u_1^2 = p_2 - p_1,$$

$$u_1 = -\sqrt{\frac{p_2 - p_1}{\rho_1}\frac{X}{X-1}} = v_1 - v_s.$$

Note we have taken the negative solution since our shock is faster than both gases, thus in the shock frame both velocities are negative.

$$v_s = v_1 + \sqrt{\frac{p_2 - p_1}{\rho_1}\frac{X}{X-1}}. \tag{3}$$

Now to relate the shock velocity, v_s, to the shocked gas velocity, v_2, we use the jump conditions and the frame boosting equations. Solving each frame boost equations for v_s, then setting equal to each other

$$v_1 - u_1 = v_2 - u_2,$$

then using (1) to replace u_1 and solve for v_2

$$v_2 = \frac{1}{X}v_1 + \frac{X-1}{X}v_s. \tag{4}$$

Now using (3) to replace v_s in (4) and simplifying, we can solve for the condition the wind must meet as

$$v_w \geq v_2 = v_1 + \sqrt{\frac{p_2 - p_1}{\rho_1}\frac{X-1}{X}}. \tag{5}$$

A Case of Equality

Here we will consider the case that the wind is exactly the shocked gas velocity. This is a finely tuned occurrence, that the wind generating a shock has the same velocity of the gas that is shocked. In the next section we will examine the more general case that the wind is faster than the shocked gas.

When $v_w = v_2$, as given in (5), the wind and shocked gas are flowing at the same speed. Thus there is no net particle flux through their interface. This means that the pressure difference at the interface is zero. However, it is allowable to have a density, and as a result also a temperature, difference. This is called a contact discontinuity.

In this finely tuned problem the velocity and pressure of the shocked gas is the same of that of the wind. However the densities need not be the same. In fact the density of the shocked gas is purely determined by the pressure ratio of the shocked gas, or in this case similarly the wind's pressure, to the ambient pressure and the density of the ambient. This is simply an application of the jump conditions given by (1).

Putting all of this together we can completely describe the space, given we know the ambient and wind. As already specified

$$p_2 = p_w,$$

$$v_2 = v_w.$$

Knowing p_2 we can now calculate X from (1), and get

$$\rho_2 = X\rho_1.$$

Lastly, we can also tell how fast the shock is propagating from (3) with our knowledge of p_2 and X.

Back Shocked

When the wind is moving faster than the shocked gas, we will have the wind smashing into the shock. This will in turn generate another shock! To clarify, the first shock (or front shock) will be a result of the wind smashing into the ambient, generating a shocked gas that will propagate through the medium. Now this shocked gas is moving slower than the wind, so the wind will smash into the shocked gas, creating another shock on the backend of the shocked gas.

The condition that sets the back shock is that the back shocked gas has the same velocity as the front shocked gas. This again is a contact discontinuity. I am failing to generate a good physical reason why this must be true. Perhaps if the back shock was faster than the front shock it would smash into the front shock and slow down, but if it was slower it would drift apart from the front shock and the wind would smash it back together.

We shall repeat much of what we did in the 'Wind Power(ed)', but now considering a shock moving slower than both gases. To start we will consider the wind/back shock interface. We will denote these with subscripts 'w' for wind and '3' for the back shocked gas. The problem is almost the same as before, but now we know the right side of the shock interface, and the shock is propagating slower than the two gas velocities. If it were faster than the back shocked gas then it would collide with the front shocked gas, and if it was faster than the wind it would no longer be a wind driven shock.

As before we will use the second Rankie-Hugoniot relation and jump conditions to solve for the shocked gas velocity. Rewriting the notation for this problem, and using 'Y' to denote the jump condition ratio at this interface (whereas 'X' refers to the front shock)

$$\frac{\rho_3}{\rho_w} = \frac{u_w}{u_3} = \frac{(\gamma+1)p_3 + (\gamma-1)p_w}{(\gamma+1)p_w + (\gamma-1)p_3} \equiv Y,$$

$$\rho_3 u_3^2 + p_3 = \rho_w u_w^2 + p_w.$$

Note that we wanted the ratio to be always greater than or equal to one, as it was for 'X'. This means it is the downstream density over the upstream density. It is incorrect to think of it as left or right side of the interface since you could just turn your head upside down and would get the wrong ratio! Thus in this case where the shock propagates relatively slow it is the opposite sides compared to the relatively fast propagating shock (assuming we keep our heads with the same orientation when we talk about the two).

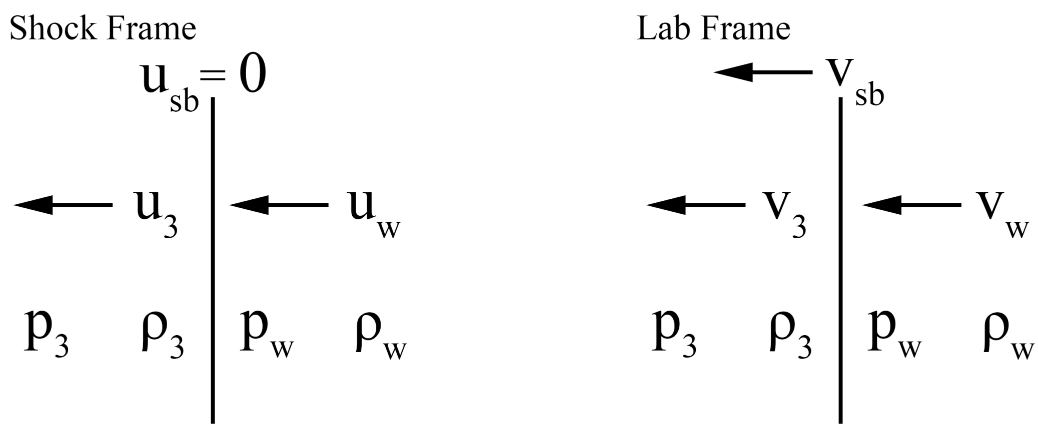

Figure 2: Left is the schematics for the shock frame, and the right for the lab frame. The arrows denote the sign convention. In this case the shock is slower than both the shocked gas and wind, which is physically sound.

Again, along the way we will solve for the velocity of propagation of the shock (denoted as v_{sb} to avoid confusion with the front shock velocity v_s). The boost equations from the lab frame to shock frame are as follows

$$u_3 = v_3 - v_{sb},$$

$$u_w = v_w - v_{sb}.$$

Following the same mathematical layout as before

$$\rho_3 u_3^2 + p_3 = \rho_w u_w^2 + p_w,$$

$$Y \rho_w \left(\frac{1}{Y} u_w\right)^2 + p_3 = \rho_w u_w^2 + p_w,$$

$$\left(1 - \frac{1}{Y}\right) \rho_w u_w^2 = p_3 - p_w,$$

$$u_w = \sqrt{\frac{p_3 - p_w}{\rho_w} \frac{Y}{Y-1}} = v_w - v_{sb}.$$

Note we have taken the positive solution since our shock is slower than both gases, thus in the shock frame both velocities are positive. Therefore,

$$v_{sb} = v_w - \sqrt{\frac{p_3 - p_w}{\rho_w}\frac{Y}{Y-1}}. \tag{6}$$

Beating a dead horse

$$v_w - u_w = v_3 - u_3,$$

ergo

$$v_3 = \frac{1}{Y}v_w + \frac{Y-1}{Y}v_{sb},$$

$$v_3 = v_w - \sqrt{\frac{p_3 - p_w}{\rho_w}\frac{Y-1}{Y}}. \tag{7}$$

However, at this point we are not done as we were before since we do not know p_3 and therefore also Y. What saves us is that we already stipulated that $p_3 = p_2$ and $v_2 = v_3$. Therefore we take (5) and set it equal to (7), using the $v_2 = v_3$ condition and then set $p_3 = p_2$, and solve for p_2. Before trying to solve it we have

$$v_1 - \sqrt{\frac{p_2 - p_1}{\rho_1}\frac{X-1}{X}} = v_w - \sqrt{\frac{p_2 - p_w}{\rho_w}\frac{Y-1}{Y}}. \tag{8}$$

Note that 'X' and 'Y' are determined by knowns and p_2, or equivalently p_3. Therefore this equation has only one unknown, namely the pressure in the shocked gas. Thus from (8) we can determine the pressures in both shocks, then we can determine the ratio that determines the jump conditions. With this we have all the knowledge about each domain of the space.

Half expectedly, we take the ambient and wind parameters as knowns. First solve (8) for the pressure in '2' and '3', then compute 'X' and 'Y', which will give us the densities in '2' and '3'. For the velocity of the shocked gas we then use (5) and (7), and the speed at which the shock propagate are given by (3) and (6). Thus the problem is solved, in theory.

"We talking about practice"

Unfortunately to me, solving (8) in complete generality is no straightforward task. Therefore in practice describing the whole space is challenging if we cannot solve for the linchpin of the problem. It should also be noted that perhaps (8) looks unsimplified since we have added variables 'X' and 'Y', when they are just functions of already present variables. First we will rid ourself of these variables and see the true expression underneath. This task will actually simply the shock propagation velocities nicely; however, it only seems to complicate our expression we wished to solve in the first place. We will then look into iteratively solving the expression, finding it rather nicely convergences.

Let's start by rewriting the terms involving out jump condition ratio.

$$\frac{X-1}{X} = 1 - \frac{1}{X} = 1 - \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma+1)p_2 + (\gamma-1)p_1} = \frac{[(\gamma+1)-(\gamma-1)]p_2 + [(\gamma-1)-(\gamma+1)]p_1}{(\gamma+1)p_2 + (\gamma-1)p_1},$$

simplified to,

$$\frac{X-1}{X} = \frac{2(p_2 - p_1)}{(\gamma+1)p_2 + (\gamma-1)p_1}.$$

Replacing $p_1 \to p_w$ and $p_2 \to p_3$, we get the similar expression involving 'Y'

$$\frac{Y-1}{Y} = \frac{2(p_3 - p_w)}{(\gamma+1)p_3 + (\gamma-1)p_w}.$$

Plugging the inverse of these into (3) and (6), we get the slightly simpler expressions

$$v_s = v_1 - \sqrt{\frac{(\gamma+1)p_2 + (\gamma-1)p_1}{2\rho_1}}, \tag{9}$$

$$v_{sb} = v_w - \sqrt{\frac{(\gamma+1)p_3 + (\gamma-1)p_w}{2\rho_w}}. \tag{10}$$

However, plugging them into (8) we get (recall $p_2 = p_3$)

$$v_1 - \sqrt{\frac{2(p_2 - p_1)^2}{\rho_1[(\gamma+1)p_2 + (\gamma-1)p_1]}} = v_w - \sqrt{\frac{2(p_2 - p_w)^2}{\rho_w[(\gamma+1)p_2 + (\gamma-1)p_w]}}.$$

Abandoning hope of an analytic solution to this expression (after minimal effort), we will now embark on numerically solving this equation, however we will find it convenient to use (8).

Iterative Pressure

Solving this expression will involve an interactive solving method. We already stated that the values of the jump condition ratio is always greater than or equal to one. The limit will be taking the pressure ratio to be infinite, this is what is called a strong shock. The strong shock limit is the maximum ratio the jump conditions can reach—for $\gamma = 5/3$ it's 4.
To solve (8) iteratively, we will assume the jump condition ratios, use them to solve for the shocked gas pressure, and then update our guess for the jump conditions. We will repeat the process until the answer converges to some shocked pressure, which will be our answer! I should note the reason for using (8) is because it's the one that Mathematica could solve for p_2 in terms of everything else. It is a mess and I will not write it out here instead find our create a mathematica notebook that does it for you (not an immediate computation on a 2013 MacBook Pro).

In general the strong shock limit is

$$\lim_{p_2/p_1 \to \infty} X = \frac{\gamma+1}{\gamma-1}.$$

The limit is the same for 'Y', just that we take the ratio that $p_3/p_w \to \infty$. This will be our first guess for the jump condition ratio. Provided below is a table of ratios and pressure at each iteration.

Ambient and wind parameters are given in the table below. They are more or less meaningless variables, half hazardily assigned.

Ambient	Wind
$p_1 = 0.0672$	$p_w = 3.0$
$\rho_1 = 0.75$	$\rho_w = 1.0$
$v_1 = 0.0$	$v_w = 3.939$

Table 1: Input parameters for simulation.

The following table is what the values of 'X', 'Y', and p_2 are at each iterative step, i.

i	X	Y	p_2
0	4.	$\frac{4000003}{1000012}$	6.00685
1	3.83938	1.50095	7.51914
2	3.87057	1.69457	7.11558
3	3.8635	1.6459	7.20019
4	3.86504	1.65627	7.18138
5	3.8647	1.65398	7.18551
6	3.86478	1.65448	7.1846
7	3.86476	1.65437	7.1848
8	3.86476	1.65439	7.18476
9	3.86476	1.65439	7.18477

This predicts $p_2 = 7.18477$, $\rho_2 = 2.89857$, $\rho_3 = 1.65439$ and $v_2 = 2.65227$. From an ATHENA simulation we find excellent agreement with these values! Here is a snapshot of the data, which you can eyeball the agreement, printing the actual data would be too much and i'm too lazy to calculate the average value in these regions.

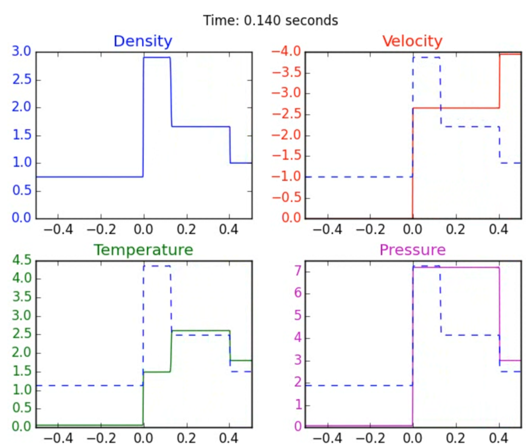

Figure 3: ATHENA simulation output at $t = .14$ for conditions given in Table 1

Ugly Details

Okay here is that equation I said I wouldn't give...

$$\begin{aligned}p_2 = &-2\sqrt{(X-1)(Y-1)(Y\rho_w(X\rho_1(v_1-v_w)^2+(X-1)(p_1-p_w))-X(Y-1)\rho_1(\rho_1-\rho_w))} \\ &p_w X^2(Y-1)^2\rho_1^2 + XY(Y-1)\rho_1\rho_w(X\rho_1(v_1-v_w)^2-(X-1)(p_1+p_w)) \\ &+Y^2(X-1)\rho_w^2(p_1(X-1)+X\rho_1(v_1-v_w)^2) \\ &* \frac{|XY\rho_1\rho_w(v_1-v_w)|}{(X(Y-1)\rho_1-XY\rho_w+Y\rho_w)^2}\end{aligned} \tag{11}$$