Piecewise Parabolic Method (PPM) of Colella & Woodward (J. Comput. Phys. 1984)

Let's solve a linear advection equation of the form

$$\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial \xi} = 0.$$

For a given time t^n, the zone average for each j^{th} zone is given by

$$a_j^n = \frac{1}{\Delta\xi_j}\int_{\xi_{j-1/2}}^{\xi_{j+1/2}} a(\xi, t^n)\,\mathrm{d}\xi.$$

Where,

$$\Delta\xi_j \equiv \xi_{j+1/2} - \xi_{j-1/2}.$$

To reconstruct $a(\xi)$ in each cell for that given time, we interpolate with a parabolic piecewise continuous function. It will be convenient to change variables ($\xi \to x$) in each cell, such that x runs from 0 to 1, thus is defined as

$$x \equiv \frac{\xi - \xi_{j-1/2}}{\Delta\xi_j}.$$

Therefore our reconstructed a in each cell is given by

$$a(x) = a_0 + a_1 x + a_2 x^2,$$

With coefficients to be determined. We can solve for two of the coefficients by defining the edge values of each cell as

$$a(0) = a_{L,j},$$
$$a(1) = a_{R,j}.$$

This results in

$$a_0 = a_{L,j},$$
$$a_1 = \Delta a_j - a_2,$$
$$\Delta a_j \equiv a_{R,j} - a_{L,j}.$$

Now our interpolated function must also satisfy the condition that when it is averaged over the whole zone it is equal to the predefined zone average, a_j^n.

$$a_j^n = \int_0^1 a(x)\,\mathrm{d}x = a_{L,j} + \frac{\Delta a_j - a_2}{2} + \frac{a_2}{3}.$$

Solving for a_2 we get

$$a_2 = -6(a_j^n - \frac{1}{2}(a_{R,j} + a_{L,j})) \equiv -a_{6,j}.$$

Therefore

$$\begin{aligned} a(x) &= a_{L,j} + (\Delta a_j + a_{6,j})x - a_{6,j}x^2, \\ &= a_{L,j} + x(\Delta a_j + a_{6,j}(1-x)). \end{aligned}$$

However, we have just defined what the left and right edge values are, but were never given them. To calculate them we