SO(4) BF

In the first order formulation of gravity. We can write gravity as a constraint SO(4) BF theory.

BF theory is a topological field theory which depends on a group G.

It depends on a choice of a Lie algebra valued 2-form field Σ and a Lie algebra valued connection A.

$$S = \int \mathrm{T}r(\Sigma \wedge F(A))$$

The eom implies that the connection is flat

$$\mathrm{d}_A \Sigma = 0 \qquad F(A) = 0$$

if one reduces to the gauge group to be SO(4) (Riemannian gravity) we can obtain gravity by imposing the simplicity constraints

It says that the two form field is not arbitrary but can be written as a wedge product

$$\Sigma^{IJ} = \left(\star(e \wedge e) + \frac{1}{\gamma} e \wedge e\right)^{IJ}$$

Immirzi parameter

$$\star\Sigma^{IJ} \equiv \frac{1}{2}\epsilon^{IJKL}\Sigma_{KL}$$ duality

γ does not affect the dynamics of GR

$$\int e_I \wedge e_J \wedge F^{IJ} = \int e_I \wedge d_A^2 e^I = \int d_A e^I \wedge d_A e_I \sim 0$$

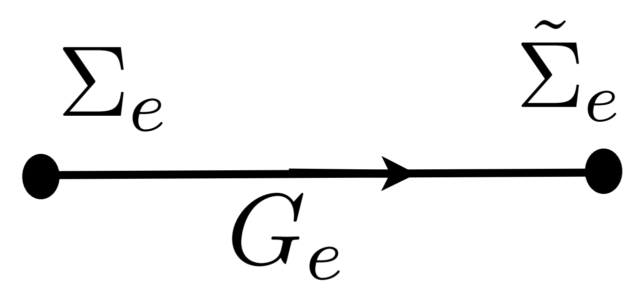
(Σ, A) are conjugate variables

At the discrete level we can describe the phase space of SO(4) BF theory in terms of a product of T^*SO(4)

To each edge of the boundary graph we assign

$$\tilde{\Sigma}_e \quad \Sigma_e \qquad \in Lie(\mathrm{SO}(4))$$

$$G_e \qquad \in \mathrm{SO}(4)$$



with relation $\Sigma_e = -G_e \cdot \tilde{\Sigma}_e$

simplicity constraints:

A bivector is-simple that is is of the form

iff
$$\Sigma = \star(U \wedge \tilde{U}) + \frac{1}{\gamma}(U \wedge \tilde{U})$$

$$U^I(\Sigma - \gamma \star \Sigma)_{IJ} = 0$$

In the time gauge $U = U^{(0)}$ with $U^{(0)} = (1,0,0,0)$

this reads

$$\underbrace{\Sigma_{0i}}_{\text{boost}} - \gamma \underbrace{\Sigma_i}_{\text{rotation}} = 0 \qquad \Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk}$$

relation with AB connection

$$w_{IJ}\Sigma^{IJ} = (w_i + \gamma w_{0i})\Sigma^i = A_i \Sigma^i$$

simplicity constraints:

to impose the simplicity condition in SO(4) BF theory one assign to every vertex a unit vector $U_v \in \mathbb{R}^4$ and impose the condition

$$U^I_{s_e}(\Sigma_e - \gamma \star \Sigma_e)_{IJ} = 0 \qquad \Sigma_e = -G_e \cdot \tilde{\Sigma}_e$$

redefining $\hat{G} \equiv G_U^{-1} G G_{\tilde{U}}$ with $G_U \cdot U^{(0)} = U$

we can go to the the time gauge

where the simplicity conditions reads

$$\Sigma_e^{0i} = \gamma \Sigma_e^i \qquad \Sigma_e = -\hat{G}_e \cdot \tilde{\Sigma}_e$$

One can think about a 4-vector as a Unitary matrix

$$U^I \to U = U^0 1 + i U^i \sigma_i$$

This gives the identification $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$

$$(G \cdot U)^I \to g_+ U (g_-)^{-1}$$

Given a bivector $\Sigma^{IJ} \in \mathbb{R}^4 \wedge \mathbb{R}^4$ we can construct its self dual and anti self dual components to be vectors in $\mathbb{R}^3$

$$\Sigma^i_\pm \equiv [(\star \pm 1)\Sigma]^{oi} = \Sigma^i \pm \Sigma^{0i}$$

this decomposition is that it maps the SO(4) action onto an SU(2)*SU(2) action

$$(G \cdot \Sigma)_\pm = g_\pm \Sigma_\pm (g_\pm)^{-1}$$

$$\Sigma^i_\pm \equiv [(\star \pm 1)\Sigma]^{oi} = \Sigma^i \pm \Sigma^{0i}$$

In this self dual formulation the condition that the bivector is simple is the condition that

$$\Sigma^i_\pm = (1 \pm \gamma)\Sigma^i$$

In other words this reads

$$\Sigma_\pm = j_\pm N \qquad j_\pm = (1 \pm \gamma) j$$

N is a unit vector

It provides an embedding

$$\Sigma_\pm = j_\pm N \qquad j_\pm = (1 \pm \gamma) j$$

N is a unit vector

It provides an embedding $T^*SU(2)$ into $T^*SO(4)$

$$(G \cdot \Sigma) = \tilde{\Sigma} \quad \longrightarrow \quad g_\pm = n e^{\xi^\pm} \tilde{n}^{-1}$$

2 twisting angles

$\xi^+ - \xi^-$ 4D dihedral angle

From classical to quantum

So far we have focused on the classical aspects of the kinematical Hilbert space.

What does it have to do with the quantum theory?

What does it have to do with the dynamics?

We would like to advocate that there is a way to relate efficiently the classical with the quantum.
The way to do so is by the choice of coherent states.

Geometrical quantisation

Main lesson from geometrical quantisation:
A coherent state is in fact an holomorphic state, related to the choice of a complex structure on the classical phase space.

finite dimensional phase space P with

symplectic structure ω $\quad closed\ invertible\ two\ form\ on\ P$

we select a complex structure $\quad J : TP \to TP \qquad J^2 = -1$

compatible with the symplectic potential. $\omega(J(X), J(Y)) = \omega(X, Y)$

determines a metric on $\quad g(X, Y) \equiv \omega(J(X), Y)$

in coordinates $\quad J(\partial_z) = i\partial_z \qquad J(\partial_{\bar{z}}) = -i\partial_{\bar{z}}$

Geometrical quantisation

These condition imply that the metric can be derived from a potential $\omega = \partial\bar{\partial}K$

If metric is positive-definite the potential is called a Kahler potential

One then chose an holomorphic line bundle L over P

and for practical purpose a trivialisation of this line bundle

$\longrightarrow$ a state is represented by an holomorphic function $\langle\Psi|z\rangle$

The completeness of the coherent state basis $\langle w|z\rangle \sim e^{\frac{1}{\hbar}K(\bar{w},z)}$

$$\langle\Psi|\Psi\rangle = \int_P \mathrm{Pf}(\omega) e^{-\frac{1}{\hbar}K(z,\bar{z})} \langle\Psi|z\rangle\langle z|\Psi\rangle$$

choice of coherent states = choice of complex structure

The 2-sphere

simplest example: $\mathbb{C}$ $K = |z|^2$

S^2 is a complex manifold $\mathbb{CP}_1 = \mathbb{C}^2/\mathbb{C}$

Element of $\mathbb{C}^2$ are spinors $|\mathbf{z}\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$ $\langle \mathbf{z}|\mathbf{z}\rangle = |z_0| + |z_1|^2$

normalised spinor can be labelled by $\mathrm{SU}(2)$ elements

$$|n_{\mathbf{z}}\rangle = \frac{|\mathbf{z}\rangle}{\sqrt{\langle \mathbf{z}|\mathbf{z}\rangle}} = n_{\mathbf{z}}|0\rangle \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

normalised spinor determines a unit vector in

$$|n\rangle\langle n| = (1+N)/2 \qquad N = n\sigma_3\, n^{-1}$$

N determine n only up to a phase

The identity decomposition is given by

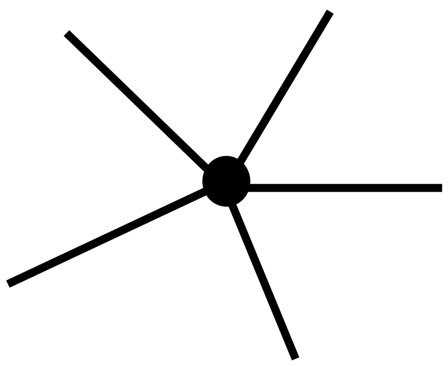
$$1_j = \mathrm{d}_j \int_{S^2} \mathrm{d}n (|n\rangle\langle n|)^{\otimes j} \qquad \mathrm{d}_j = 2j+1$$

These states represent states of minimal uncertainty picked around the classical vector

$$X = jN = jn\tau_3 n^{-1}$$

Intertwinner

Given a vertex carrying spins $j_1, \cdots j_N$



we can define a coherent intertwinner by averaging over the group

$$|\vec{\jmath}, n_i\rangle \equiv \int_{\mathrm{SU}(2)} \mathrm{d}g \, (g|n_1\rangle^{\otimes j_1} \otimes \cdots \otimes g|n_N\rangle^{\otimes j_N})$$

These states satisfy the closure relation $\sum_i \hat{J}_i |\vec{\jmath}, n_i\rangle = 0$

However their label do not necessarily $\sum_i j_i N_i \neq 0$

Intertwinner

Hopefully Quantisation commute with reduction [Q,R]=0

We can restrict the label of the coherent state to be in the Polyhedral space

$$P_{\vec{\jmath}} = \{n_i | \sum_i j_i N_i = 0\}$$

and still satisfy the unity decomposition

$$1_{\vec{\jmath}} = \int_{P_{\vec{\jmath}}} \mu_{\vec{\jmath}}(n_i) |\vec{\jmath}, n_i\rangle\langle\vec{\jmath}, n_i|$$

integration over framed polytopes

The prefactor is

$$\mu_{\vec{\jmath}} \sim \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g \, e^{K_{\vec{\jmath}}(gn_i)} \mathrm{Pf}(\omega)(gn_i) \quad \sim \omega_{P_{\vec{\jmath}}}$$

$$K_{\vec{\jmath}}(|\mathbf{z}_i\rangle) = \sum_i (2j_i + 1) \ln(\langle \mathbf{z}_i | \mathbf{z}_i \rangle)$$

Kahler potential

Intertwinner

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integration over framed polytopes

It follows from the Guillemin-Sternberg isomorphism

$$P//G = P^*/G^{\mathbb{C}}$$

The coherent spin network states are labelleb by twisted geometries

SO(4) states

we can extend our construction to SO(4) simple coherent states.

at the classical level a simple bivector

$$\Sigma = \star(U \wedge \tilde{U}) + \frac{1}{\gamma}(U \wedge \tilde{U})$$

with Immirzi parameter

is given by a pair

$$\Sigma_u = (u_+, u_-)(j_+ N, j_- N)$$

Rotation mapping the reference vector to U

$$U = u_+ u_-^{-1}$$

$$j_\pm = (1 \pm \gamma) j$$

satisfying the condition $\qquad u_\pm N u_\pm^{-1} = \tilde{u}_\pm \tilde{N} \tilde{u}_\pm^{-1}$

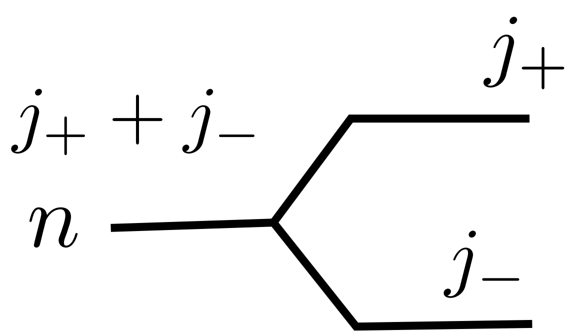
SO(4) states

we can extend our construction to SO(4) simple coherent states.

at the quantum level a simple bivector is repesented by a map

$$\Sigma_\gamma : V_j \rightarrow V_{j_+} \otimes V_{j_-}$$

$$|n\rangle^{\otimes j} \rightarrow |n\rangle^{\otimes j_+} \otimes |n\rangle^{\otimes j_-}$$



with $\quad j_+ + j_- = 2j \qquad j_+ - j_- = 2\gamma j$

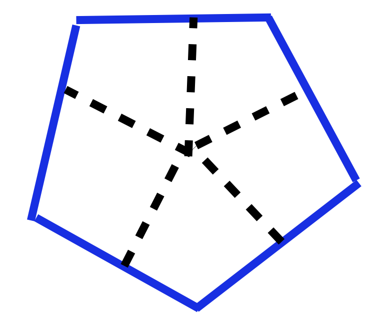
$$u_\pm N u_\pm^{-1} = \tilde{u}_\pm \tilde{N} \tilde{u}_\pm^{-1}$$

BF amplitudes

we can finally write down the amplitudes

For SU(2) BF theory the quantum amplitudes can be written in terms of a spin foam model.

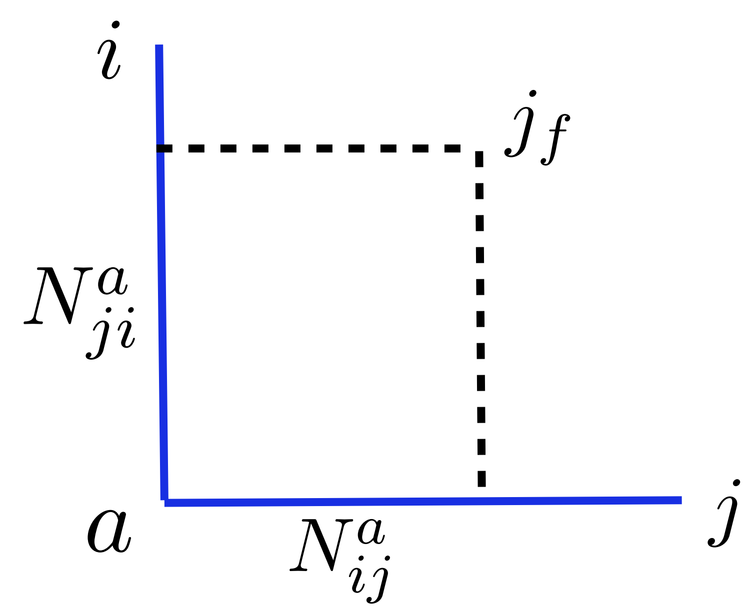
To any face of the 2D complex we assign a spin



to any wedge $w = (iaj)$ where i, a, j are three vertices we assign the two vectors

we also assign a group element per edge

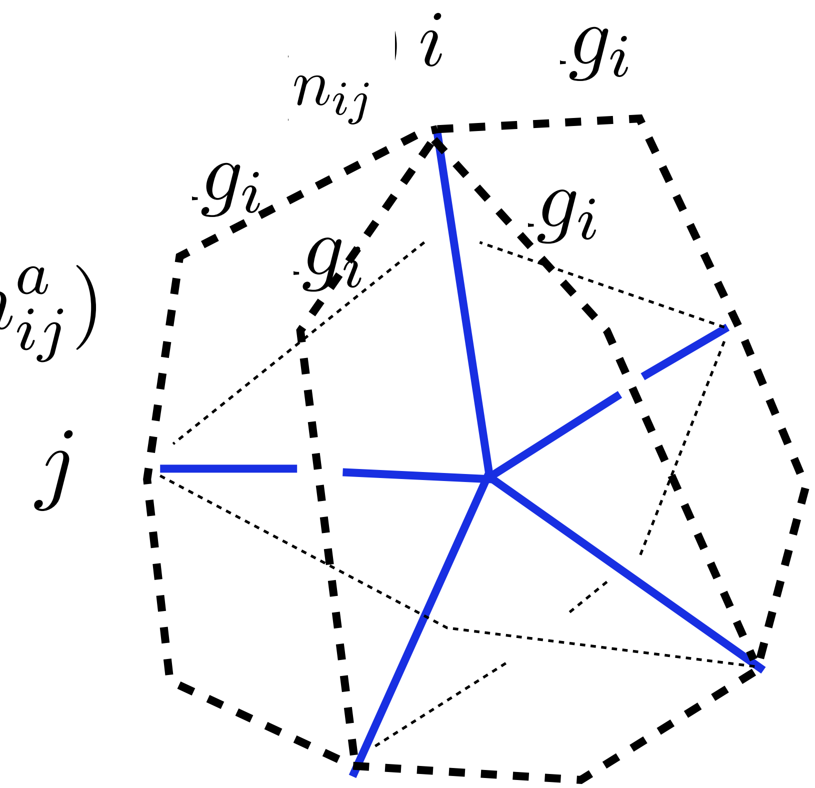
Any slicing inherites a spin and 2 vector per edge



BF amplitudes

The amplitude can be written as a product of amplitude for every vertex

$$Z_S(\Gamma, j, n) = \sum_{j_f} \mathrm{d}_{j_f} \int \prod_{w=(aij)} \mathrm{d}_{j_w} \mathrm{d}n_w \prod_a A_a(j^a_{ij}, n^a_{ij})$$



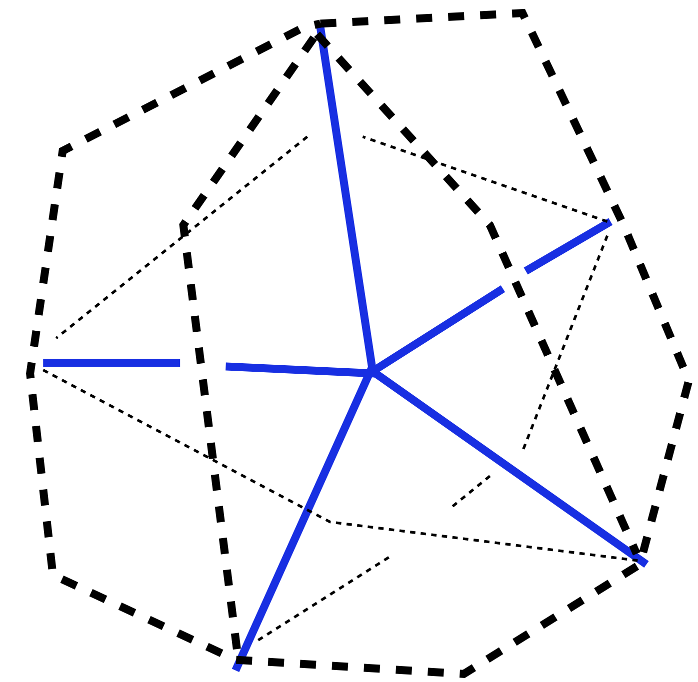
$$A_v(j_{ij}, n_{ij}) = \int_{\mathrm{SU}(2)} \prod_i \mathrm{d}g_i \prod_{i<j} \langle n_{ij} | g_i^{-1} g_j | n_{ji} \rangle^{j_{ij}}$$

One group element per internal edge

BF amplitudes

Proof: after integrating over n_w we obtain a product over each face of the type

$$\chi_{j_f}\left(g_{a_1 a_2} \cdots g_{a_n a_1}\right) \qquad g_{a_1 a_2} = g^{a_1}_{a_2} (g^{a_2}_{a_1})^{-1}$$



Holonomy along face

The sum over $\hat{f}$ impose the flatness of the discrete connection.

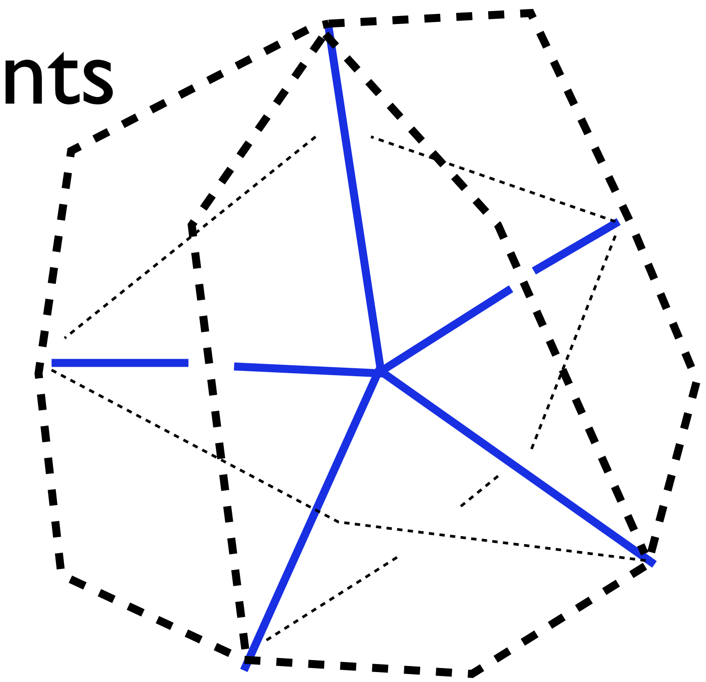
$$\sum_{j_f} \mathrm{d}_{j_f} \chi_{j_f}(G) = \delta(G)$$

The amplitude is the projection over flat connection

Gravity amplitudes

According to our analysis the 4d Riemannian gravity amplitude is obtained by imposing the simplicity constraints of the SO(4) BF theory

after imposition of the simplicity constraints the vertex amplitude is



$$A_v^{(\gamma)}(j_{ij}, n_{ij}) = A_v(\gamma_+ j_{ij}, n_{ij}) A_v(\gamma_- j_{ij}, n_{ij})$$

$$A_v(j_{ij}, n_{ij}) = \int_{\mathrm{SU}(2)} \prod_i \mathrm{d}g_i \prod_{i<j} \langle n_{ij} | g_i^{-1} g_j | n_{ji} \rangle^{j_{ij}}$$

$$\frac{\gamma_+}{\gamma_-} = \frac{1+\gamma}{1-\gamma}$$

Gravity amplitudes

What does this have to do with gravity?

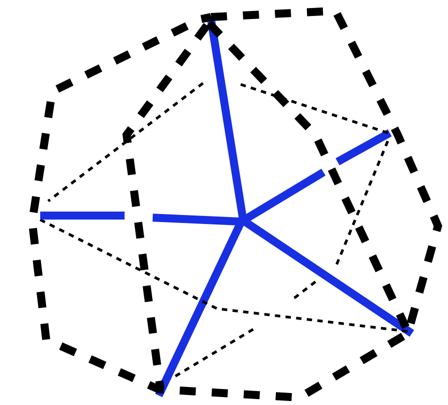
We look at the behavior of the amplitude in the semi-classical limit

Since the area are given by the spins in unit of the Planck lenght

$$A_f = \gamma \ell_P^2 j_f$$

The semi-classical limit for fixed area corresponds to the limit where all the spins uniformely go to infinity.

semi-classical limit



Lets assume that the 2d complex is 5 valent. Each edge of the spin foam corresponds then to a tetrahedra

Lets look at the amplitude where we integrate out everything but the spins

$$Z_S(\Gamma, j, n) \equiv \int \prod_{w=(aij)} \mathrm{d}n_w \prod_a A_a(j_{ij}^{+a}, n_{ij}^a) A_a(j_{ij}^{-a}, n_{ij}^a)$$

arbitrary complex

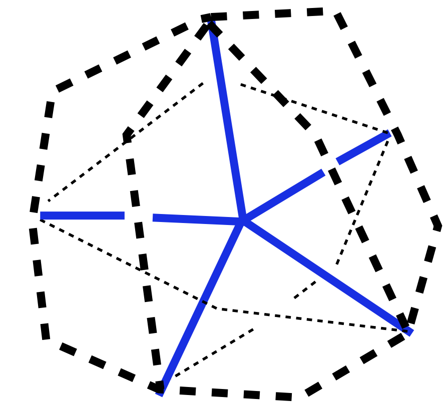
restrict the integration to geometrical tetrahedron

We need to compute the asymptotic of the vertex amplitude

$$A_v(j_{ij}, n_{ij}) = \int_{\mathrm{SU}(2)} \prod_i \mathrm{d}g_i^+ \mathrm{d}g_i^- \prod_{i<j} \langle n_{ij} | (g_i^+)^{-1} g_j^+ | n_{ji} \rangle^{j_{ij}^+} \langle n_{ij} | (g_i^-)^{-1} g_j^- | n_{ji} \rangle^{j_{ij}^-}$$

We assume non degeneracy by restricting the integration to the sector where $\det(U_i) \neq 0$ $\quad U_i = (g_i^+)^{-1} (g_i^-)^{-1}$

semi-classical limit



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The main result: If there exists a set of edge length of the triangulation dual to spin foam such that $j_f = A_f(\ell_e)$

Then Z is asymptotic to the exponential of the Regge action

$$Z(\Gamma, j, n) \sim e^{iS(\ell_e)} + c.c$$

If there is not Z is exponentially suppressed

$$A_v(j_{ij}, n_{ij}) = \int_{\mathrm{SU}(2)} \prod_i \mathrm{d}g_i^+ \mathrm{d}g_i^- \prod_{i<j} \langle n_{ij}|(g_i^+)^{-1} g_j^+ |n_{ji}\rangle^{j_{ij}^+} \langle n_{ij}|(g_i^-)^{-1} g_j^- |n_{ji}\rangle^{j_{ij}^-}$$

First we can restrict the boundary labels to satisfy the closure condition

$$\sum_j j_{ij} N_{ij} = 0$$

Second the amplitude is exponentially supressed unless

$$\langle n_{ij}|g_i^{-1} g_j|n_{ji}\rangle \breve{=} 1$$

i-e unless $g_j^{\pm}|n_{ji}\rangle = e^{\frac{i}{2}(\nu_{ij} \pm \Delta_{ij})} g_i^{\pm}|n_{ij}\rangle$

The closure condition and this condition imply that the variation of the phase of the weight is stationary

therefore

$$A_v(j_{ij}, n_{ij}) = \int_{\mathrm{SU}(2)} \prod_i \mathrm{d}g_i^+ \mathrm{d}g_i^- \prod_{i<j} \langle n_{ij}|(g_i^+)^{-1} g_j^+ |n_{ji}\rangle^{j_{ij}^+} \langle n_{ij}|(g_i^-)^{-1} g_j^- |n_{ji}\rangle^{j_{ij}^-}$$

By the stationary phase approximation the amplitude is given by the evaluation of the weight on the solutions of

$$g_i^{\pm}|n_{ij}\rangle = e^{\frac{i}{2}(\nu_{ji} \pm \Delta_{ij})} g_j^{\pm}|n_{ij}\rangle$$

Due to non degeneracy condition there are 2 solutions $g_i^{\pm} = u_i^{\mp}$

or $g_i^{\pm} = u_i^{\pm}$ where $U_i = u_i^+ (u_i^-)^{-1}$ normal of the geo 4-simplex j_{ij}

Thus $A_v(j_{ij}, n_{ij}) \sim e^{\sum_{ij} j_{ij}\nu_{ij}} \left(e^{\sum_{ij} \gamma j_{ij} \Delta_{ij}} + e^{\sum_{ij} \gamma j_{ij} \Delta_{ij}} \right)$

Regge action

Δ_{ij} is in fact the dihedral angle

$\sum_{w \subset f} \nu_w = 0\ (\pi)$ ν_{ij} define a flat connection and is therefore pure gauge

Δ_{ij} is in fact the dihedral angle

$$
\begin{aligned}
U_i \cdot U_j = \mathrm{Tr}\left(u_i^+ (u_i^-)^{-1} u_j^- (u_j^+)^{-1}\right) &= \mathrm{Tr}\left((u_i^-)^{-1} u_j^- (u_j^+)^{-1} u_i^+\right) \\
&= \langle n_{ij} | (u_i^-)^{-1} u_j^- (u_j^+)^{-1} u_i^+ | n_{ij} \rangle + c.c \\
&= e^{\frac{i}{2}(\nu_{ij} + \Delta_{ij})} \langle n_{ij} | (u_i^-)^{-1} u_j^- | n_{ji} \rangle + c.c \\
&= e^{i\Delta_{ij}} + c.c
\end{aligned}
$$

$$
u_i^{\pm} | n_{ij} \rangle = e^{\frac{i}{2}(\nu_{ji} \pm \Delta_{ij})} u_j^{\pm} | n_{ij} \rangle
$$

Conclusion

We have seen that the SU(2) spin network Hilbert space associated with a graph is a universal object that appears in many geometrical instances

In the quantization of LQG

In the quantization of discrete twisted geometry

In the spin foam quantization of Plebanski theory (constraint BF)

This leads to a simple proposal for the quantum dynamics

which is purely algebraic in nature and has beautiful semi-classical property

This can be extended to Lorentzian case

Conclusion

There are still many open problems

Some technical some more fundamental :

Insertion of a cosmological constant

Asymptotic boundary condition (Link with S-matrix, AdS/CFT)

Time evolution, coupling to matter ...

Link with the Hamiltonian formulation

Summation over spines, is GFT proposal correct?

Prove that the amplitude respect spacetime diffeomorphism

So far this is a model not a full theory...

Conclusion

We have seen that the SU(2) spin network Hilbert space associated with a graph is a universal object that appears in many geometrical instances

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