The Fluid/Gravity Correspondence

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PASI Summer School
June 30, 2010

Outline

1. Review
   - fluid dynamics
   - gravity
2. Iterative construction of bulk g_{MN} and boundary $T^{\mu\nu}$
   - Structure of Einstein equations
   - Implementation to 1st order
3. Analysis of 2nd order solution
   - boundary stress tensor and transport coefficients
   - bulk geometry and event horizon
4. Summary & Remarks

Fluid dynamics

- Fluid dynamics is continuum effective description of any microscopic QFT valid when scales of variation are long compared to mean free path ℓ_{mfp}.
- The fluid description assumes that the system achieves local thermodynamic equilibrium.

Regime of validity: "long-wavelength approximation"

For local temperature of the fluid T
and scale of variation of the dynamical degrees of freedom L,
local equilibrium demands:

$$L\,T \equiv \frac{1}{\epsilon} \gg 1$$

Fluid dynamics

Dynamical degress of freedom:

- Local temperature T
- Fluid velocity u_μ (normalized $u_\mu \, u^\mu = -1$)

$\Rightarrow$ This leaves d functions, $T(x^\mu)$ and $u_\nu(x^\mu)$, which specify our fluid configuration.

(x^μ are coordinates on the boundary spacetime on which the fluid lives.)

The conformal fluid stress tensor $T^{\mu\nu}$

Encode all the fluid information by stress tensor $T^{\mu\nu}$, which is

- Traceless: $T^\mu_\mu = 0$
- Conserved: $\nabla_\mu T^{\mu\nu} = 0$

The conservation equation captures the dynamics.

Form of conformal fluid stress tensor

Form of stress tensor is determined by symmetries,
order by order in derivative expansion;
fluid properties specified by finite # of undetermined coefficients.
Using Weyl-covariant formulation, in d dimensions

$$\begin{aligned} T^{\mu\nu} = {} & P\left(\gamma^{\mu\nu} + d\, u^\mu\, u^\nu\right) - 2\,\eta\,\sigma^{\mu\nu} \\ & + 2\,\eta\left[\tau_1\ u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} - \tau_2\left(\omega^\mu{}_\lambda\,\sigma^{\lambda\nu} + \omega^\nu{}_\lambda\,\sigma^{\lambda\mu}\right)\right] \\ & + \xi_\sigma[\sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}] + \xi_C\ C_{\mu\alpha\nu\beta}\, u^\alpha\, u^\beta \\ & + \xi_\omega\,[\omega^\mu{}_\lambda\,\omega^{\lambda\nu} + \frac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\,\omega_{\alpha\beta}] + \ldots \end{aligned}$$

where Weyl-covariance demands that

$$P \propto T^d\ , \quad \eta \propto T^{d-1}\ , \quad \tau_{1,2} \propto T^{-2}\ , \quad \xi_{\sigma,C,\omega} \propto T^{d-2}$$

Gravity in the bulk

- Consider any 2-derivative theory of 5-d gravity interacting with other fields with AdS_5 as a solution (e.g. *II*B SUGRA on $AdS_5 \times S^5$). Solution space has a universal sub-sector: pure gravity with negative cosmological constant

$$E_{MN} \equiv R_{MN} - \frac{1}{2} R \, g_{MN} + \Lambda \, g_{MN} = 0$$

$(R_{AdS} = 1 \;\Rightarrow\; \Lambda = -6)$
We will focus on this sub-sector in long-wavelength limit.

- Apart from the AdS_5 solution, there is a 4-parameter family of solutions representing asymptotically-AdS_5 boosted planar black holes.
- We will use these solutions to construct general dynamical spacetimes characterized by fluid-dynamical configurations.

0^{th} order: boosted Schwarzschild-AdS black hole

- Start with the well-known stationary solution: (derivation)
  boosted Schwarzschild-AdS_5 black hole (w/ planar symmetry)

  $$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2\, \left(\eta_{\mu\nu} + [1 - f(r/\pi T)]\, u_\mu\, u_\nu\right)\, dx^\mu dx^\nu\ ,$$

  with $f(r) \equiv 1 - \frac{1}{r^4}$
- It is parameterized by 4 parameters:
  temperature T and boosts u_i.
- The bulk black hole is dual to a bdy perfect fluid with

  $$T^{\mu\nu} = \pi^4\, T^4\, \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right)$$

Deforming the 0^{th} order solution

- Now promote parameters u_μ and T to functions depending on the boundary coordinates x^μ:

$$\begin{aligned} ds^2 &= -2\, u_\mu(x)\, dx^\mu dr \\ &+ r^2 \left(\eta_{\mu\nu} + [1 - f(r/\pi T(x))]\, u_\mu(x)\, u_\nu(x)\right) dx^\mu dx^\nu \end{aligned}$$

  Call such a metric $g^{(0)}$.

- Note: $g^{(0)}$ does NOT satisfy the equations of motion:

$$E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R - 6\, g_{MN} = 0$$

- But starting from here we will construct an iterative solution.

A perturbation scheme for gravity

Assume that the variation in local temperature and velocities are slow

$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}(\epsilon) \ , \qquad \frac{\partial_\mu u}{T} \sim \mathcal{O}(\epsilon)$$

$\Rightarrow$ In local patches the solution is like a boosted planar black hole.

Basic idea

The perturbative scheme is aimed at constructing a regular bulk solution, by patching together pieces of the uniform boosted black hole.

Use ϵ as a book-keeping parameter (counting # of x^μ derivatives), and expand:

$$g = \sum_{k=0}^{\infty} \epsilon^k \, g^{(k)} \ , \qquad T = \sum_{k=0}^{\infty} \epsilon^k \, T^{(k)} \ , \qquad u = \sum_{k=0}^{\infty} \epsilon^k \, u^{(k)}$$

A perturbation scheme for gravity

At a given order in the ϵ-expansion we find equations for $g^{(k)}$. These are ultra-local in the field theory directions and take the schematic form:

$$\mathbb{H}\left[g^{(0)}(u_\mu^{(0)}, T^{(0)})\right] g^{(k)}(x^\mu) = s_k$$

- $\mathbb{H}$ is a second order linear differential operator in r alone.
- s_k are regular source terms which are built out of $g^{(n)}$ with $n \leq k-1$.

A perturbation scheme for gravity

Importantly the equations of motion split up into two kinds:

- Constraint equations: $E_{r\mu} = 0$, which implement stress-tensor conservation (at one lower order).
- Dynamical equations: $E_{\mu\nu} = 0$ and $E_{rr} = 0$ allow determination of $g^{(k)}$.

We solve the dynamical equations

$$g^{(k)} = \text{particular}(s_k) + \text{homogeneous}(\mathbb{H})$$

subject to

- regularity in the interior
- asymptotically AdS boundary conditions

(coordinate choice for g)

Computation at first order

- To solve the equations to first order we need to ensure conservation of the perfect fluid stress tensor

$$\partial_\mu T^{\mu\nu}_{(0)} = 0$$

which needs to be solved only locally (at say $x^\mu = 0$).

- This can be used to eliminate derivatives of $T^{(0)}$ in terms of those of $u_i^{(0)}$.

$$\partial_v (\pi\, T^{(0)})^{-1} = \frac{1}{3}\, \partial_i\, u_i^{(0)} \ , \qquad \partial_i\, (\pi\, T^{(0)})^{-1} = \partial_v\, u_i^{(0)}$$

- Then we solve $\mathbb{H} g^{(1)} = s_1$ where the operators and sources are given as follows:

Computation at first order

The operator $\mathbb{H}$: Useful to decompose metric perturbations into $SO(3)$ representations: scalars **1**, vectors **3** and symmetric traceless tensors **5**. For instance, we find:

$$\mathbb{H}_{\mathbf{3}}\# = \frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr}\#\right)$$

$$\mathbb{H}_{\mathbf{5}}\# = \frac{d}{dr}\left(r^5\, f(r)\,\frac{d}{dr}\#\right)$$

The source terms: These differ at various orders in perturbation theory. At first order:

$$s_1^{\mathbf{3}} = -\frac{3}{r^2}\,\partial_v\, u_i^{(0)}$$

$$s_1^{\mathbf{5}} = -6\, r^2\, \sigma_{ij}^{(0)}$$

Explicit solution to first order

Bulk metric:

$$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2\, \left(\eta_{\mu\nu} + [1 - f(r/\pi T)]\, u_\mu\, u_\nu\right)\, dx^\mu dx^\nu$$
$$+ 2r \left[\frac{r}{\pi T}\, F(r/\pi T)\, \sigma_{\mu\nu} + \frac{1}{3}\, u_\mu u_\nu\, \partial_\lambda u^\lambda - \frac{1}{2}\, u^\lambda \partial_\lambda \left(u_\nu u_\mu\right)\right]\, dx^\mu dx^\nu,$$

with

$$F(r) = \int_r^\infty dx\, \frac{x^2 + x + 1}{x(x+1)\,(x^2+1)} = \frac{1}{4}\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right]$$

Boundary stress tensor:

$$T^{\mu\nu} = \pi^4\, T^4\, (4\, u^\mu u^\nu + \eta^{\mu\nu}) - 2\,\pi^3\, T^3\, \sigma^{\mu\nu}.$$

Note: $\frac{\eta}{s} = \frac{1}{4\pi}$ in agreement with well-known results.

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The d-dimensional conformal fluid from AdS$_{d+1}$

Similar procedure can be implemented at any order.
Generalizing to d dimensions and bdy metric $\gamma_{\mu\nu}$ [cf. 0809.4272], to second order:

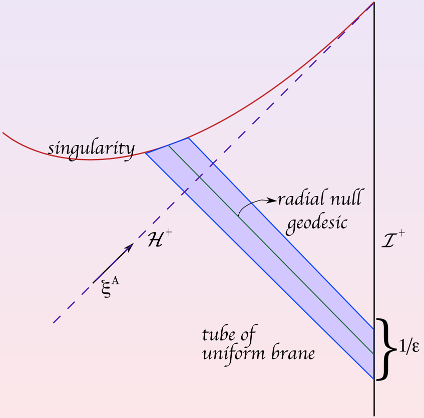
$$\begin{aligned} T_{\mu\nu} &= P\,(\gamma_{\mu\nu} + d\,u_\mu\,u_\nu) - 2\,\eta\,\sigma_{\mu\nu} \\ &- 2\,\eta\,\tau_\omega \left[u^\lambda\,\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\,\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\,\sigma_{\mu\lambda}\right] \\ &+ 2\,\eta\,b\left[u^\lambda\,\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\,\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\,\sigma^{\alpha\beta}}{d-1}\,P_{\mu\nu} + C_{\mu\alpha\nu\beta}\,u^\alpha\,u^\beta\right] \end{aligned} \tag{1}$$

$$\text{with} \qquad b \equiv \frac{d}{4\pi T} \qquad ; \qquad P = \frac{1}{16\pi\,G_{\text{AdS}}\,b^d} \qquad ;$$

$$\eta = \frac{s}{4\pi} = \frac{1}{16\pi\,G_{\text{AdS}}\,b^{d-1}} \qquad \text{and} \qquad \tau_\omega = b\int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}\,dy$$

The spacetime geometry dual to fluids

The bulk solution thus constructed is tubewise approximated by a planar black hole!



Bulk causal structure; in each "tube" metric approximates uniformly boosted Schwarzschild-AdS planar black hole.

The event horizon

The background has a regular event horizon.

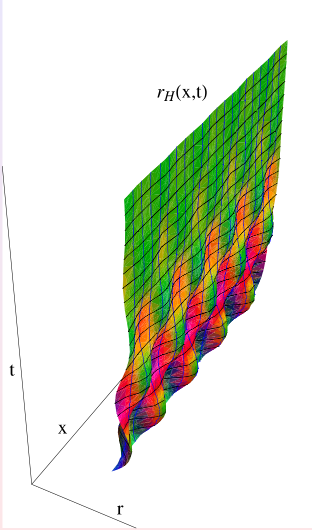
- One can determine the event horizon locally using the fact that the solution settles down at late times to an uniformly boosted planar black hole.
- The horizon location can be determined within the perturbation scheme

$$r = r_H(x) = \pi\, T(x) + \sum_{k=1}^{\infty} \epsilon^k \, r_{(k)}(x)$$

- In fact, $r_{(k)}(x)$ is determined algebraically by demanding that the surface given by $r = r_H(x)$ be null.

simpler analogy

Cartoon of the event horizon



Note:

- Horizon is null everywhere
- Late time approach to uniform planar black hole
- Horizon area increases

The Entropy current

- Given a bulk geometry with a horizon we can determine the Bekenstein-Hawking entropy.
- Bulk construction of entropy: using area-form A of spatial slices of the event horizon in Planck units.

Fluid entropy current

The area-form A on event horizon can be pulled back to the boundary to define a fluid entropy current J_S^μ

$$J_S = *_\eta A$$

with non-negative divergence

$$\partial_\mu J_S^\mu \geq 0$$

▸ Entropy details

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Properties of Entropy current

- The bulk-boundary pull-back is facilitated by our coordinates: pull-back along radial ingoing geodesics (const r)

$$x^{\mu}(\mathcal{H}) \quad \rightarrow \quad x^{\mu}(\mathrm{bdy})$$

- Fluid entropy current consistent with second law and equations of motion naively* has a 5 parameter ambiguity
  details
- Bulk construction of entropy current is also ambiguous:
  (i) ability to add total derivative terms without changing area
  (ii) pull-back is ambiguous to boundary diffeomorphisms.
  At second order this results in a two parameter ambiguity for Weyl covariant current with positive divergence.

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Summary

- $\exists$ a map between conformal fluid configurations on $\mathbb{R}^{d-1,1}$ & regular asymp. AdS_{d+1} planar non-uniform black holes
  $\Rightarrow$ gain insight into *generic* behaviour of gravity
- Bulk spacetime solutions
  - naturally uphold Cosmic Censorship
  - imply a new variant of Uniqueness Theorem
- Long-wavelength regime of fluid dynamics allows this construction to any order in a boundary derivative expansion.
- This yields *local* determination of the event horizon.
- The solutions satisfy the Area increase theorem & corresponding entropy current satisfies the 2^{nd} law.
- Recovered the well-known value of viscosity: $\frac{\eta}{s} = \frac{1}{4\pi}$
- Predicted second order transport coefficients, characterising our CFT fluid.

Generalizations

- other dimensions

  (cf. Van Raamsdonk; Haack, Yarom; Bhattacharyya, Loganayagam, Mandal, Minwalla, Sharma)
- fluids on curved manifolds

  (Cf. Bhattacharyya, Loganayagam, Minwalla, Nampuri, Trivedi, Wadia)
- include matter
  (richer dynamics, but at expense of losing universality)
  - dilaton ($\rightarrow$ induces forcing)

    (cf. Bhattacharyya, Loganayagam, Minwalla, Nampuri, Trivedi, Wadia)
  - Maxwell U(1) field ($\rightarrow$ extra conserved charge)

    (cf. Erdmenger, Haack, Kaminski, Yarom; Banerjee, Bhattacharya, Bhattacharyya, Dutta, Loganayagam, Surowka; Hur, Kim, Sin)
  - 3 Maxwell fields + 3 scalars (cf. Torabian, Yee)
  - magnetic and dyonic charges (cf. Hansen, Kraus; Caldarelli, Dias, Klemm)
- non-conformal fluids

  (cf. Kanitscheider, Skenderis; David, Mahato, Wadia)
- non-ralativistic fluids

  (cf. Rangamani, Ross, Son, Thompson; Bhattacharyya, Minwalla, Wadia)

Puzzles & future directions

- Role of non-long-wavelength bulk semiclassical solutions
- More detailed bulk analysis: horizon topology, nature of curvature singularity, Cosmic Censorship
- $\exists$ striking difference between turbulence in 3+1 and 2+1 nonrelativistic fluids (eg. inverse cascade)
  $\overset{?}{\Longrightarrow}$ qualitative difference in gravitational dynamics (eg. equilibration time of AdS_4 vs. AdS_5 BHs) Van Raamsdonk
- Relation to the black hole Membrane Paradigm ▸ comments
- Generalizations: extremal fluids (superfluids), confining theories (domain walls), ...
- Finite N effects
- Gravity dual of turbulence

Boosted Schwarzschild-AdS black hole

- Static Schwarzschild-AdS black hole in planar limit:

$$ds^2 = r^2 \left(-f(r)\, dt^2 + \sum_i (dx^i)^2 \right) + \frac{dr^2}{r^2\, f(r)}$$

with $f(r) = 1 - \frac{r_+^4}{r^4}$. ($\rightsquigarrow$ temperature $T = r_+/\pi$.)

- To avoid coordinate singularity at the horizon $r = r_+$, use ingoing coordinates: $v = t + r_*$ where $dr_* = \frac{dr}{r^2\, f(r)}$:

$$ds^2 = -r^2\, f(r)\, dv^2 + 2\, dv\, dr + r^2 \sum_i (dx^i)^2$$

- Now 'covariantize' by boosting: $v \to u_\mu\, x^\mu$, $x_i \to P_{i\mu}\, x^\mu$.

back

Choice of coordinates

For dealing with regularity issues etc., it is simplest to work in an analog of ingoing Eddington-Finkelstein coordinates.

$$ds^2 = -2\, u_\mu(x)\, \mathcal{S}(r,x)\, dr\, dx^\mu + \chi_{\mu\nu}(r,x)\, dx^\mu\, dx^\nu$$

- The choice of coordinates is such that $x^\mu = \textit{constant}$ are ingoing null geodesics.
- It is well adapted to discuss features of horizon, such as entropy in the fluid language.

back

Event horizon in Vaidya-AdS

Vaidya = spher. sym. black hole with ingoing null matter:

$$ds^2 = -\left(1 - \frac{2\,m(v)}{r}\right) dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$

- suppose horizon is at $r = r_H(v)$
- normal $n = dr - \dot{r}\,dv$ is null when

$$r_H(v) = 2\,m(v) + 2\,r_H(v)\,\dot{r}_H(v)$$

- Exact solution gives horizon *non-locally* in terms of $m(v)$.
- But for $m(v)$ slowly varying, $\dot{m}(v) = \mathcal{O}(\epsilon)$, $m\,\ddot{m} = \mathcal{O}(\epsilon^2)$, use ansatz

$$r_H = 2\,m + a\,m\,\dot{m} + b\,m\,\dot{m}^2 + c\,m^2\,\ddot{m} + \dots$$

- Iterative solution gives $a = 8,\ b = 64,\ c = 32,\ \dots$

back

Expression for entropy current

The gravitational entropy current:

$$(4\pi\,\eta)^{-1}\,J_S^\mu = \left[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\right]u^\mu$$
$$+\, b^2\left[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right]$$
$$+\, C_1\;b\;\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \ldots$$

with

$$A_1 = \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8}$$
$$B_1 = \frac{1}{4}; \qquad B_2 = \frac{1}{2}$$
$$C_1 = C_2 = 0$$

◂ Entropy current

Divergence of entropy current

Entropy current:

$$(4\pi\,\eta)^{-1}\,J_S^\mu = \left[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\right]\,u^\mu$$
$$+\, b^2\left[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right]$$
$$+\, C_1\;b\;\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \ldots$$

Divergence of entropy current:

$$4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu = \frac{b}{2}\left[\sigma_{\mu\nu} + b\left(2\,A_1 + 4\,A_3 - \frac{1}{2} + \frac{1}{4}\ln 2\right)u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu}\right.$$
$$\left. +4\,b\,(A_2 + A_3)\,\omega^{\mu\alpha}\omega_\alpha{}^\nu + b\,(4\,A_3 - \frac{1}{2})(\sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu) + b\,C_2\,\mathcal{D}^\mu\ell^\nu\right]^2$$
$$+\,(B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu\mathcal{D}_\lambda\sigma^{\mu\lambda} + \ldots$$

Non-negativity of divergence: $\mathcal{D}_\mu\,J_S^\mu \geq 0$ (when $\sigma^{\mu\nu} = 0$) demands

$$B_1 = 2\,A_3\;, \qquad C_1 + C_2 = 0$$

Horizon physics described by fluid dynamics...

Where does the fluid live?

- On the event horizon?
  (null hypersurface, defined globally...)
- On the dynamical horizon? — Gourgoulhon & Jaramillo
  (spacelike hypersurface)
- On the stretched horizon? (a la Membrane Paradigm)

Membrane Paradigm — Thorne, Macdonald, Price

Horizon interpreted as a fluid membrane with certain dissipative properties: (e.g. electrical conductivity, shear & bulk viscosity, etc.)

- On the spacetime boundary. (AdS/CFT)
  Fluid dynamics describes the full spacetime, not just horizon.

back

Weyl covariant formalism

Consider conformal transformations of metric and fluid variables:

$$g_{\mu\nu} = e^{2\phi(x)}\tilde{g}_{\mu\nu} \quad , \qquad u_\mu = e^{\phi(x)}\tilde{u}_\mu \quad , \qquad T = e^{-\phi(x)}\,\tilde{T} \tag{2}$$

To construct Weyl-covariant derivative, use the 'gauge field'

$$\mathcal{A}_\nu \equiv u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\nu = \tilde{\mathcal{A}}_\nu + \partial_\nu \phi. \tag{3}$$

Then for tensor $Q^{\mu\ldots}_{\nu\ldots}$ of weight w

$$\begin{aligned} \mathcal{D}_\lambda \; Q^{\mu\ldots}_{\nu\ldots} \equiv \nabla_\lambda \; Q^{\mu\ldots}_{\nu\ldots} &+ w \; \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots} \\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots \\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda \mathcal{A}_\nu - \delta^\alpha_\nu \mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots \end{aligned} \tag{4}$$

is also Weyl-covariant with weight w.

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Kerr-AdS BH in fluid variables

Weyl-covariant formalism allows to express rotating AdS black holes more elegantly and intuitively in fluid variables:

$$ds^2 = -2u_\mu dx^\mu \left(dr + r\,\mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda} u^\lambda - \omega_\mu{}^\lambda \omega_{\lambda\nu}\right] dx^\mu dx^\nu + \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\,\delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu \tag{5}$$

with

$$\mathcal{A}_\nu \equiv u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\nu \tag{6}$$

and Weyl-covariantized Schouten tensor

$$\mathcal{S}_{\mu\nu} \equiv \frac{1}{d-2}\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R} g_{\mu\nu}}{2(d-1)}\right) \tag{7}$$