Black Holes and Thermodynamics

I: Classical Black Holes

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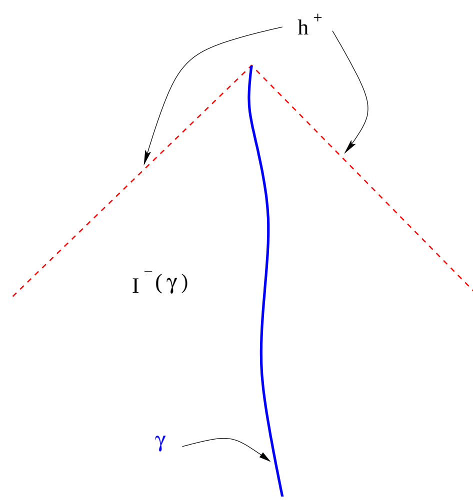
General references: R.M. Wald *General Relativity* University of Chicago Press (Chicago, 1984); R.M. Wald Living Rev. Rel. **4**, 6 (2001).

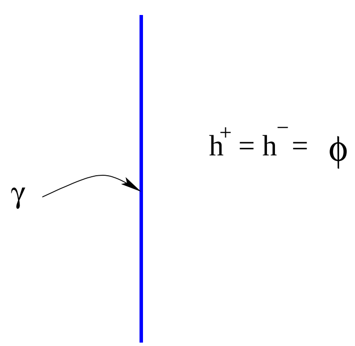
Horizons

An observer in a spacetime (M, g_{ab}) is represented by an inextendible timelike curve γ. Let $I^-(\gamma)$ denote the chronological past of γ. The future horizon, h^+, of γ is defined to be the boundary, $\dot{I}^-(\gamma)$ of $I^-(\gamma)$.

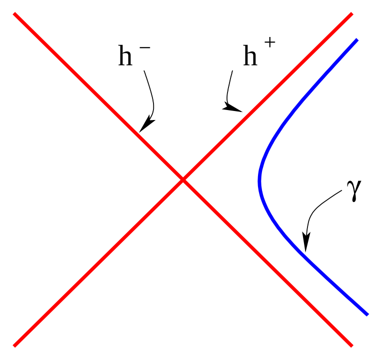


Theorem: Each point $p \in h^+$ lies on a null geodesic segment contained entirely within h^+ that is future inextendible. Furthermore, the convergence of these null geodesics that generate h^+ cannot become infinite at a point on h^+.

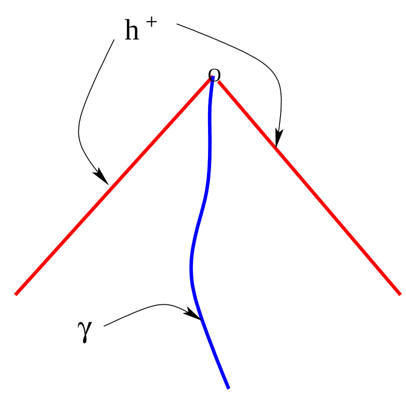
Can similarly define a past horizon, h^-. Can also define h^+ and h^- for families of observers.



Inertial Observer in Minkowski Spacetime



Uniformly Accelerating Observer in Minkowski Spacetime



Observer in Minkowski Spacetime with a point removed

Black Holes and Event Horizons

Consider an asymptotically flat spacetime (M, g_{ab}). (The notion of asymptotic flatness can be defined precisely using the notion of conformal null infinity.) Consider the family of observers Γ who escape to arbitrarily large distances at late times. If the past of these observers $I^-(\Gamma)$ fails to be the entire spacetime, then a black hole $B \equiv M - I^-(\Gamma)$ is said to be present. The horizon, h^+, of these observers is called the future event horizon of the black hole.

This definition allows "naked singularities" to be present.

Cosmic Censorship

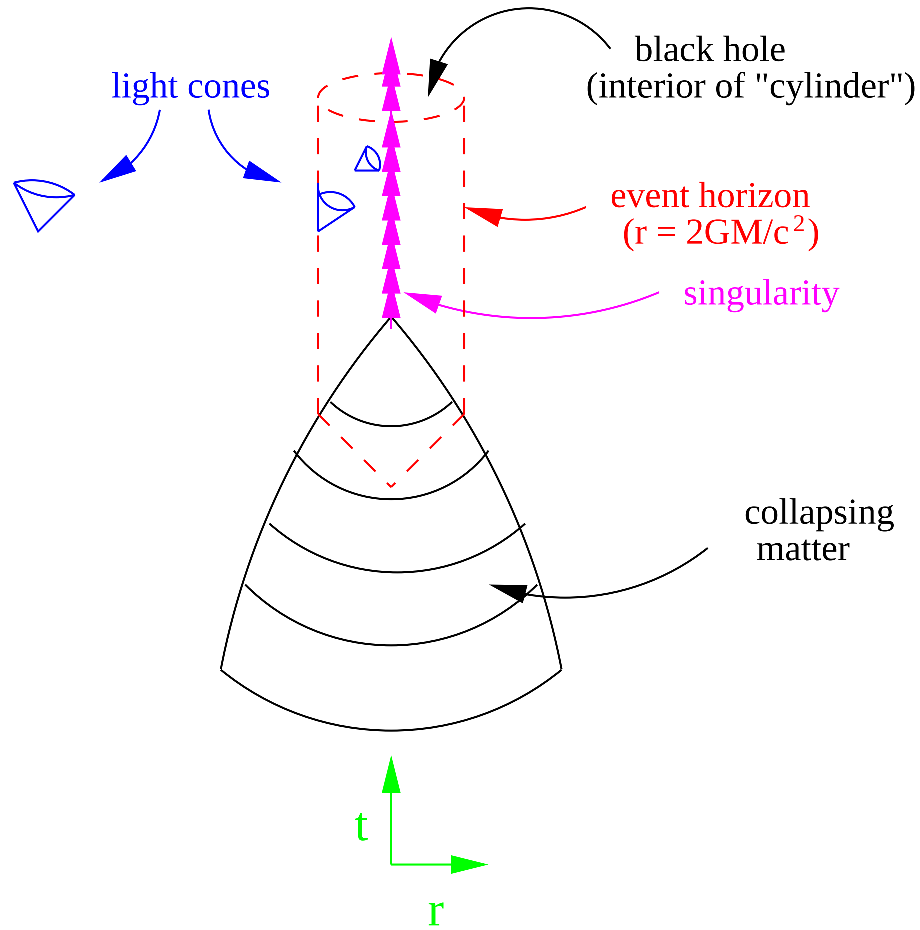
A Cauchy surface, $\mathcal{C}$, in a (time orientable) spacetime (M, g_{ab}) is a set with the property that every inextendible timelike curve in M intersects $\mathcal{C}$ in precisely one point. (M, g_{ab}) is said to be globally hyperbolic if it possesses a Cauchy surface $\mathcal{C}$. This implies that M has topology $\mathbf{R} \times \mathcal{C}$.

An asymptotically flat spacetime (M, g_{ab}) possessing a black hole is said to be predictable if there exists a region of M containing the entire exterior region and the event horizon, h^+, that is globally hyperbolic. This expresses the idea that no "naked singularities" are present.

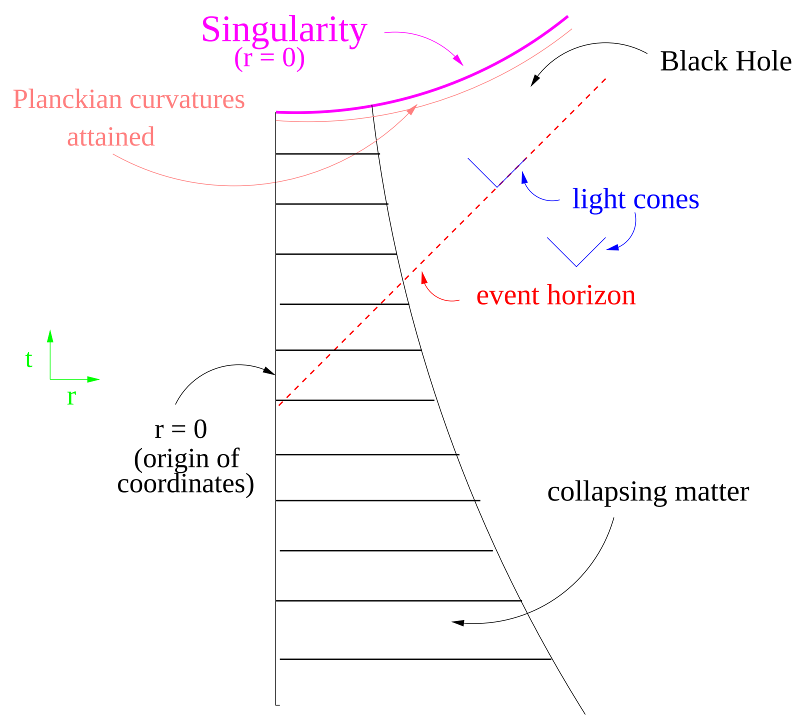
Cosmic Censor Hypothesis: The maximal Cauchy evolution—which is automatically globally hyperbolic—of an asymptotically flat initial data set (with suitable matter fields) generically yields an asymptotically flat spacetime with complete null infinity.

The validity of the cosmic censor hypothesis would assure that any observer who stays outside of black holes could not be causally influenced by singularities.

Spacetime Diagram of Gravitational Collapse



Spacetime Diagram of Gravitational Collapse with Angular Directions Suppressed and Light Cones "Straightened Out"



Null Geodesics and the Raychauduri Equation

For a congruence of null geodesics with affine parameter λ and null tangent k^a, define the expansion, θ, by

$$\theta = \nabla_a k^a$$

The area, A of an infinitesimal area element transported along the null geodesics varies as

$$\frac{d(\ln A)}{d\lambda} = \theta$$

For null geodesics that generate a null hypersurface (such as the event horizon of a black hole), the twist, ω_{ab}, vanishes. The Raychauduri equation—which is a direct consequence of the geodesic deviation equation—then yields

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b$$

where σ_{ab} is the shear of the congruence. Thus, provided that $R_{ab}k^a k^b \geq 0$ (i.e., the null energy condition holds), we have

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$$

which implies

$$\frac{1}{\theta(\lambda)} \leq \frac{1}{\theta_0} + \frac{1}{2}\lambda$$

Consequently, if $\theta_0 < 0$, then $\theta(\lambda_1) = -\infty$ at some $\lambda_1 < 2/|\theta_0|$ (provided that the geodesic can be extended that far).

The Area Theorem

Any horizon h^+, is generated by future inextendible null geodesics; cannot have $\theta = -\infty$ at any point of h^+. Thus, if the horizon generators are complete, must have $\theta \geq 0$. However, for a predictable black hole, can show that $\theta \geq 0$ without having to assume that the generators of the event horizon are future complete—by a clever argument involving deforming the horizon outwards at a point where $\theta < 0$.

Let S_1 be a Cauchy surface for the globally hyperbolic region appearing in the definition of predictable black hole. Let S_2 be another Cauchy surface lying to the future of S_1. Since the generators of h^+ are future complete, all of the generators of h^+ at S_1 also are present at S_2. Since $\theta \geq 0$, it follows that the area carried by the generators of h^+ at S_2 is greater or equal to $A[S_1 \cap h^+]$. In addition, new horizon generators may be present at S_2. Thus, $A[S_2 \cap h^+] \geq A[S_1 \cap h^+]$, i.e., we have the following theorem:

Area Theorem: For a predictable black hole with $R_{ab}k^a k^b \geq 0$, the surface area A of the event horizon h^+ never decreases with time.

Killing Vector Fields

An isometry is a diffeomorphism ("coordinate transformation") that leaves the metric, g_{ab} invariant. A Killing vector field, ξ^a, is the infinitesimal generator of a one-parameter group of isometries. It satisfies

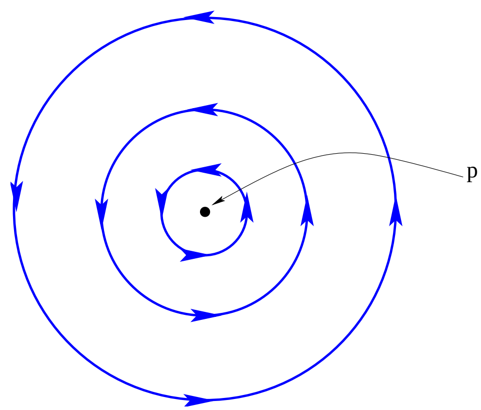
$$0 = \mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}$$

For a Killing field ξ^a, let $F_{ab} = \nabla_a \xi_b = \nabla_{[a}\xi_{b]}$. Then ξ^a is uniquely determined by its value and the value of F_{ab} at an aribitrarily chosen single point p.

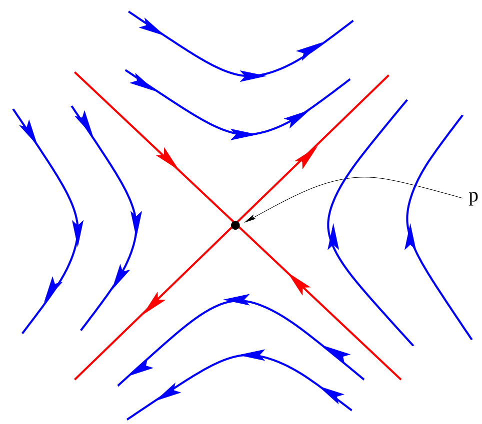
Bifurcate Killing Horizons

2-dimensions: Suppose a Killing field ξ^a vanishes at a point p. Then ξ^a is determined by F_{ab} at p. In 2-dimensions, $F_{ab} = \propto \epsilon_{ab}$, so ξ^a is unique up to scaling

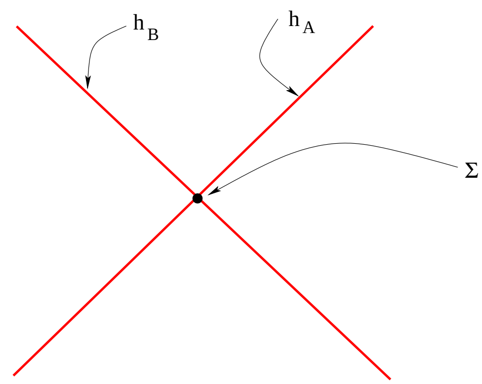
If g_{ab} is Riemannian, the orbits of the isometries generated by ξ^a must be closed and, near p, the orbit structure is like a rotation in flat space:



Similarly, if g_{ab} is Lorentzian, the isometries must carry the null geodesics through p into themselves and, near p, the orbit structure is like a Lorentz boost in 2-dimensional Minkowski spacetime:



4-dimensions: Similar results to the 2-dimensional case hold if ξ^a vanishes on a 2-dimensional surface Σ. In particular, if g_{ab} is Lorentzian and Σ is spacelike, then, near Σ, the orbit structure of ξ^a will look like a Lorentz boost in 4-dimensional Minkowski spacetime. The pair of intersecting (at Σ) null surfaces h_A and h_B generated by the null geodesics orthogonal to Σ is called a bifurcate Killing horizon.



It follows that ξ^a is normal to both h_A and h_B. More generally, any null surface h having the property that a Killing field is normal to it is called a Killing horizon.

Surface Gravity and the Zeroth Law

Let h be a Killing horizon associated with Killing field ξ^a. Let U denote an affine parameterization of the null geodesic generators of h and let k^a denote the corresponding tangent. Since ξ^a is normal to h, we have

$$\xi^a = f k^a$$

where $f = \partial U/\partial u$ where u denotes the Killing parameter along the null generators of h. Define the surface gravity, κ, of h by

$$\kappa = \xi^a \nabla_a \ln f = \partial \ln f/\partial u$$

Equivalently, we have $\xi^b \nabla_b \xi^a = \kappa \xi^a$ on h. It follows immediately that κ is constant along each generator of h.

Consequently, the relationship between affine parameter U and Killing parameter u on an arbitrary Killing horizon is given by

$$U = \exp(\kappa u)$$

Can also show that

$$\kappa = \lim_{h} (Va)$$

where $V \equiv [-\xi^a \xi_a]^{1/2}$ is the "redshift factor" and a is the proper acceleration of observers following orbits of ξ^a.

In general, κ can vary from generator to generator of h. However, we have the following three theorems:

Zeroth Law (1st version): Let h be a (connected) Killing horizon in a spacetime in which Einstein's equation holds with matter satisfying the dominant energy condition. Then κ is constant on h.

Zeroth Law (2nd version): Let h be a (connected) Killing horizon. Suppose that either (i) ξ^a is hypersurface orthogonal (static case) or (ii) there exists a second Killing field ψ^a which commutes with ξ^a and satisfies $\nabla_a(\psi^b \omega_b) = 0$ on h, where ω_a is the twist of ξ^a (stationary-axisymmetric case with "t-ϕ reflection symmetry"). Then κ is constant on h.

Zeroth Law (3rd version): Let h_A and h_B be the two null surfaces comprising a (connected) bifurcate Killing horizon. Then κ is constant on h_A and h_B.

Constancy of κ and Bifurcate Killing Horizons

As just stated, κ is constant over a bifurcate Killing horizon. Conversely, it can be shown that if κ is constant and non-zero over a Killing horizon h, then h can be extended locally (if necessary) so that it is one of the null surfaces (i.e., h_A or h_B) of a bifurcate Killing horizon.

In view of the first version of the 0th law, we see that apart from "degenerate horizons" (i.e., horizons with $\kappa = 0$), bifurcate horizons should be the only types of Killing horizons relevant to general relativity.

Event Horizons and Killing Horizons

Hawking Rigidity Theorem: Let (M, g_{ab}) be a stationary, asymptotically flat solution of Einstein's equation (with matter satisfying suitable hyperbolic equations) that contains a black hole. Then the event horizon, h^+, of the black hole is a Killing horizon.

The stationary Killing field, ξ^a, must be tangent to h^+. If ξ^a is normal to h^+ (so that h^+ is a Killing horizon of ξ^a), then it can be shown that ξ^a is hypersurface orhogonal, i.e., the spacetime is static. If ξ^a is not normal to h^+, then there must exist another Killing field, χ^a , that is normal to the horizon. It can then be further shown that there is a linear combination, ψ^a, of ξ^a and χ^a whose orbits are spacelike and closed, i.e., the spacetime is axisymmetric. Thus, a stationary black hole must be static or axisymmetric.

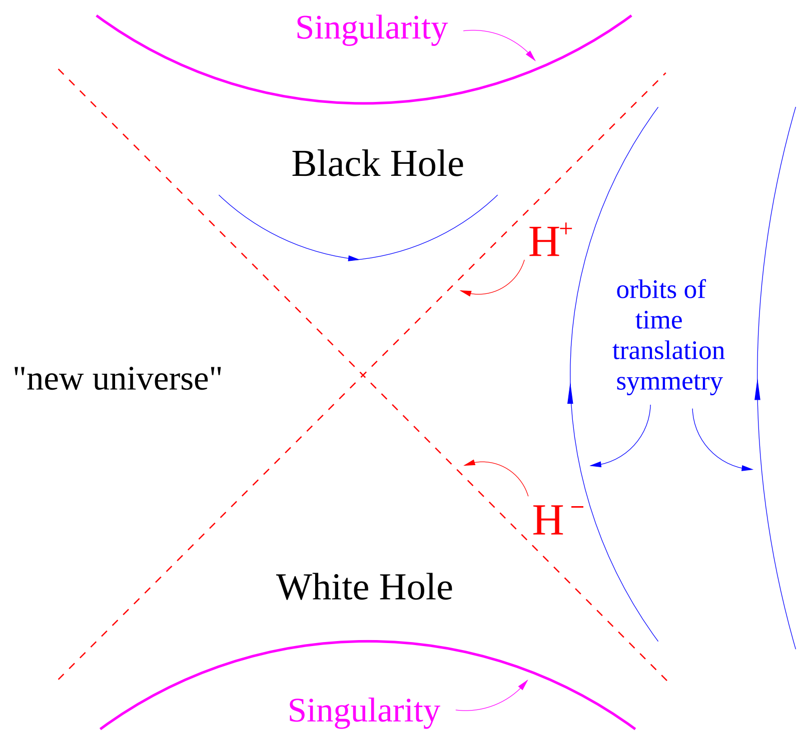
We can choose the normalization of χ^a so that

$$\chi^a = \xi^a + \Omega\psi^a$$

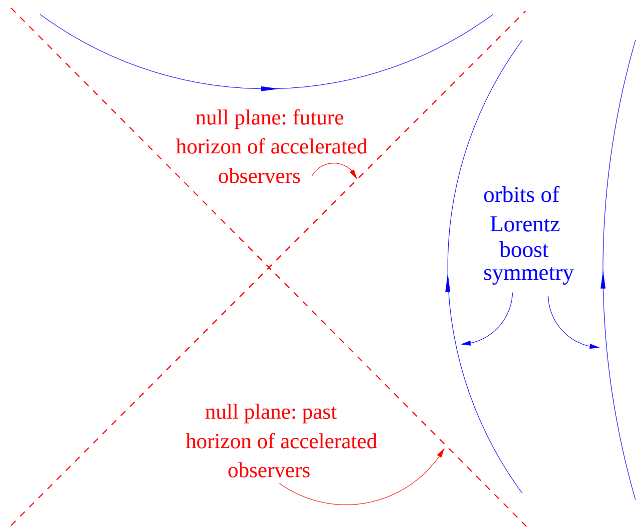
where Ω is a constant, called the angular velocity of the horizon.

Idealized ("Analytically Continued") Black Hole

"Equilibrium State"



A Close Analog: Lorentz Boosts in Minkowski Spacetime



Note: For a black hole with $M \sim 10^9 M_\odot$, the curvature at the horizon of the black hole is smaller than the curvature in this room! An observer falling into such a black hole would hardly be able to tell from local measurements that he/she is not in Minkowski spacetime.

Summary

- If cosmic censorship holds, then—starting with nonsingular initial conditions—gravitational collapse will result in a predictable black hole.
- The surface area of the event horizon of a black hole will be non-decreasing with time (2nd law).

It is natural to expect that, once formed, a black hole will quickly asymptotically approach a stationary ("equilibrium") final state. The event horizon of this stationary final state black hole:

- will be a Killing horizon
- will have constant surface gravity, κ (0th law)

- if $\kappa \neq 0$, will have bifurcate Killing horizon structure