Lecture #10: Using The Residue Theorem

I. More Examples

II. Integrals of Trig Functions

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I. More Examples

Last time we were talking about finding Residues. The coefficient $c_n$ in $f(z) = \sum_{n=0}^{\infty} c_n z^n$.

Let's do just a few more examples to warm up. Then we'll use the residue theorem to evaluate some interesting integrals.

1) Find all residues of

$$f(z) = \frac{z+2}{z(z-1)(z-3)}$$

Simple poles at $z = 0, 1/2, 3$

$\text{Res}(0) = \frac{z+2}{z(z-1)(z-3)} \bigg|_{z=0} = \frac{2}{(-1)(-3)} = \frac{1}{3}$

$\text{Res}(1/2) = \frac{z+2}{z(z-1)(z-3)} \bigg|_{z=1/2} = \frac{\frac{5}{2}}{\frac{1}{2}(1/2-3)} = -1$

Because $(z-1/2) = 2(z-3/2)$

$\text{Res}(3) = \frac{z+2}{z(z-1)(z-3)} \bigg|_{z=3} = \frac{5}{3(2\cdot3-1)} = \frac{1}{3}$

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\[
\oint_{C_1} f(z) = 2\pi i \left( \frac{2}{3} - 1 + \frac{1}{3} \right) = 0
\]

\[
\oint_{C_2} f(z) = 2\pi i \text{ Res}(0) = \frac{4\pi i}{3}
\]

\[
\oint_{C_3} f(z) = -2\pi i \left( -1 + \frac{1}{3} \right) = \frac{4\pi i}{3}
\]
\( f(z) = \frac{\cos(2z)}{z(2z + 5)} \)  
5, -1 poles at 0 4 -5

\[
\text{Res}_{z=0} = \frac{\cos(2\cdot0)}{5} = \frac{1}{5}, \quad \text{Res}_{z=-5} = \frac{\cos(-10)}{-5} = -\frac{1}{5} \cos(10)
\]

\[
\int_{C} f(z) \, dz = 2\pi i \left( \text{Res}_{z=0} - \text{Res}_{z=-5} \right)
\]

\[
= -\frac{2\pi i}{5} \left( \cos(10) - 1 \right)
\]

3. What is \( \text{Res}_{z=0} \) if \( f(z) = \frac{1}{\sin z} \)?

In general if \( f(z) = \frac{g(z)}{h(z)} \) with zero at \( z_0 \)

\[ \Rightarrow \text{Simple pole} \]

\[
\Rightarrow \text{Res}(z_0) = \lim_{z \to z_0} (z - z_0) g(z)
\]

\[
= g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)} \geq g(z_0) \frac{1}{h'(z_0)}
\]

by L'Hopital's rule

\[
\Rightarrow \text{Res}(z_0) = \frac{1}{\cos(z_0)} = 1
\]

\[
\text{Res}_{z=(-n)} = \frac{1}{\cos(n\pi)} = (-1)^n
\]

\[
\int_{C} \frac{1}{\sin z} \, dz = 2\pi i \left( \sum_{n=0}^{\infty} \text{Res}_{z=n\pi} \right) = 2\pi i \cdot \text{Res}(3\pi)
\]

\[\text{cancel in pairs}\]

\[\Rightarrow -2\pi i
\]

\[3\pi < 0 < 4\pi\]
II. Truncareln Integals

We have seen that the residue theorem often makes it easy to evaluate closed contour integrals in the complex plane. However, integrals in physics are usually evaluated along the real line with contours that are not closed.

Nevertheless, the residue theorem can be very useful! This is perhaps easiest to explain by looking at some examples.

**Examples**

1. Evaluate

\[ I = \int_0^{2\pi} \frac{\cos \theta \, d\theta}{5 + 4 \cos \theta} \]

Although this is a real integral, we can write it in terms of a contour integral in \( C \) by defining \( z = e^{i\theta} \).

Then \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2} (z + \frac{1}{z}) \)

\[ \Rightarrow \quad d\theta = i e^{i\theta} \, dz = iz \, dz \quad \text{and} \quad C \]

\[ I = \oint_C \left( \frac{1}{2} \left( z + \frac{1}{z} \right) \right) \frac{dz}{z(5 + \frac{4}{z} + \frac{4}{z^2})} \times \left( \frac{2}{z^2} \right) \]

\[ = -\frac{i}{2} \oint_C \frac{z^2 + 1}{z(2z^2 + 5z + 2)} \, dz \]

Simple poles at \( z = 0 \)

\[ z = -\frac{5 \pm \sqrt{25 - 16}}{4} = -\frac{5 \pm 3}{4} \]

\[ \Rightarrow \quad z = \frac{-8}{4} = -2 \]

\[ \text{and} \quad z = \frac{-2}{4} = -\frac{1}{2} \]

\[ I = -\frac{i}{2} \oint_C \frac{z^2 + 1}{z(2z + 1)(z + 2)} \, dz \]
Contour has \( |z|=1 \Rightarrow \) encloses \( n \) poles at \( 0, -1/2 \)

Let \( f(z) = \frac{z^{2}+1}{(z+1/2)(z+2)} \)

\[
\text{Res}_{z=0} f(z) = \lim_{z \to 0} \frac{z^{2}+1}{z+1/2} = \lim_{z \to 0} \frac{z}{1/2} = \frac{1}{2} \cdot 2 = 1
\]

\[
\text{Res}_{z=-1/2} f(z) = \lim_{z \to -1/2} \frac{z^{2}+1}{z+2} = \lim_{z \to -1/2} \frac{1}{2} = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{3}
\]

So \( I \approx \frac{1}{4} (2\pi \chi) \left(1-\frac{5}{3}\right) = \frac{\pi}{2} (-2/3) = -\pi/3 \)

Note that the final result is real, as it must be since we started with an integral of a real function over a real interval.

Note also that the procedure was quite straightforward. The same trick can be used on any rational function of \( \sin \) & \( \cos \) so long as the denominator does not vanish for some value \( z \).

In contrast, you might try doing the above integral by elementary methods.... good luck!

2. \( I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \)

This integral is easy to calculate via elementary methods:

\[
I = \tan^{-1} x \bigg|_{-\infty}^{\infty} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi
\]

But what about \( I_n = \int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}} \) for \( n \in \mathbb{Z}^+ \)?

These can be done by using a simple technique that is useful for a wide class of integrals:

\[
\text{write } I_n = \int_{0}^{\infty} \frac{dz}{c_0 + z^{2n}}
\]
Note: \( \frac{1}{1+2\pi n} \) vanishes only when \( z^{2n} = -1 \) 
or \( z^n = \pm i \)

So, \( \frac{1}{1+2\pi n} \) is analytic on \( C_0 \).

But! \( C_0 \) is not closed. If it were, we could use the residue theorem to evaluate the result.

Question: How much does the answer change if we close the contour?

Consider \( C_R \),

\[
\int_{C_R} \frac{dz}{1+z^{2n}} = \int_{-R}^{R} \frac{dx}{1+x^{2n}} + \int_{0}^{\pi} \frac{iR e^{i\theta} d\theta}{1+R^{2n} e^{2i\theta}}
\]

So, \( I_n = \lim_{R \to \infty} \oint_{C_R} \frac{dz}{1+z^{2n}} \to 0 \) as \( R \to \infty \).

Now \( \frac{1}{1+2\pi n} \) is analytic except where \( z^{2n} = -1 = \exp(i\pi + 2\pi in) \)

\[
\Rightarrow z = e^{i\pi \left( \frac{1+2\pi n}{2n} \right)}
\]

In general, in simple poles, \( n \) of which are inside \( C_R \)

for large \( R \).
\[ n = 1 \]
\[ \frac{1}{1 + z} = \frac{1}{\frac{1}{2} + z} \]
\[ \frac{1}{\frac{1}{2} + i} \mid_{n = 1} = \frac{1}{2^2} = \frac{-i}{2} \]

\[ I_1 = \pi \text{ i } \left( \frac{-1}{\pi} \right) = \pi \quad \checkmark \]

\[ n = 2 \]
\[ \frac{1}{1 + z^2} = \frac{1}{2 - e^{i\pi/4}} \quad \frac{1}{2 - e^{i\pi/4}} \quad \frac{1}{2 - e^{i3\pi/4}} \quad \frac{1}{2 - e^{i3\pi/4}} \quad \text{poles} \]

\[ \rho_2(z_0) = \frac{\text{Res}(z_0)}{z \to z_0} \frac{z - z_0}{1 + z^2} = \frac{1}{d \left( z \right)(1 + z^2)} \mid_{z = z_0} = \frac{1}{4 \pi i} \mid_{z = z_0} \]

\[ \rho_2 \left( e^{i\pi/4} \right) = \frac{1}{4} e^{-3i\pi/4} \]

\[ \rho_2 \left( e^{i3\pi/4} \right) = \frac{1}{4} e^{9i\pi/4} = \frac{1}{4} e^{i\pi/4} (e^{i\pi/4})^2 \]

\[ \ast \]

\[ e^{i\pi} = -1 \quad \text{so} \quad z_0^2 = -\frac{1}{z_0} \quad \text{i.e.} \quad z_0^{-3} = -\frac{1}{z_0} \]

\[ S_0 \quad I_2 = 2 \pi \text{ i } \left( -\frac{1}{4} \right) \]

\[ \left( \text{sum of roots } \frac{1}{z} = -1 \right) \]

\[ \text{in upper half-plane} \]

\[ \frac{1}{\sqrt{2} \left( 1 + \frac{1}{\sqrt{2}} \right)} + \frac{1}{\sqrt{2} \left( \frac{1}{\sqrt{2}} - 1 \right)} = \frac{2\sqrt{2}}{\sqrt{2}} \]

\[ = (2\pi \text{ i } \left( -\frac{1}{2} \right) \left( \frac{2\sqrt{2}}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}} \quad \text{real} \]

Use same tricks to do \( I_n \). At a pole \( z_0 = e^{i\pi} \left( \frac{\sqrt{2}}{2\pi} \right) \approx \frac{1}{1 + z^n} \)

\[ \rho_2(z_0) = \frac{\text{Res}(z_0)}{z \to z_0} \frac{z - z_0}{1 + z^n} = \frac{1}{d \left( z \right)(1 + z^n)} \mid_{z = z_0} = \frac{1}{2n \left( z_0 \right)^{n-1}} \]

\[ = \frac{1}{2n} \frac{-z_0}{z_0^{n-1}} = \frac{-z_0}{2n} \]

\[ I_n = \pi \text{ i } \left( -\frac{1}{z_n} \right) \]

\[ \left( \text{sum of roots } \frac{z^n - 1}{z^n} \right) \]

\[ \text{in upper half-plane} \]

\[ \omega = \frac{2\pi i}{z_n} \left( \frac{\sqrt{2}}{2\pi} \right) \]
\[ = e^{\frac{i\pi}{ln}} \sum_{n=0}^{n-1} \left( e^{\frac{i\pi/n}{n}} \right)^n \]

\[ = e^{i\pi/n} \frac{1 - e^{-i\pi/n}}{1 - e^{i\pi/n}} \quad \text{Note: } e^{-i\pi} = -1 \]

\[ = e^{i\pi/n} \frac{2}{1 - e^{i\pi/n}} = \frac{2}{e^{-i\pi/n} - 1} \]

\[ = \frac{2}{e^{-i\pi/n} - e^{i\pi/n}} = -\frac{1}{n} \frac{2i}{e^{i\pi/n} - e^{-i\pi/n}} = \frac{\pi}{\sin(\pi/n)} \]

So \[ I_n = \left( \frac{i\pi}{n} \right) \left( \frac{1}{\sin(\pi/n)} \right) = \frac{\pi}{n} \frac{1}{\sin(\pi/n)} \]

Check: \[ I_1 = \frac{\pi}{\sin(\pi)} = 0/1 = 0 \quad \checkmark \]

\[ I_2 = \frac{\pi/2}{\sin(\pi/2)} = \frac{\pi}{2} \sqrt{2} \quad \checkmark \]

Note: For the above examples we could equally well have closed the contour in the lower half-plane. Doing so yields the same answer.