Lecture 11: More Integrals & Contours

I. More Integral Examples

II. Jordan's Lemma

I. More Integral Examples

Last time we saw that it can be useful to take a contour along the real axis and to add extra bits to close it in the complex plane.

For example, a contour along the real axis might be closed in either the upper or lower half-plane.

In our example from last time, either choice was ok, but some cases are more subtle.

Example:

\[ I = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)} \, dx \]

First, can we close the contour using \( 1 / 2 \) or \( 1 / 2 \) ?

We can see the contour using \( 1 / 2 \) or \( 1 / 2 \), where does \( e^{ix} \) behave for large imaginary \( x \)?

\[ e^{ix} = \frac{e^{i2} + e^{-i2}}{2} = e^{(ix) + (ix)} + e^{-(ix) - (ix)} = e^{ix - iy} + e^{-ix + iy} \]

\[ \rightarrow \infty \text{ as } y \rightarrow \infty \]

So, \( i(1 + 1) \) appears large for either \( \infty \) or \( -\infty \).

Note that \( e^{iy} + e^{-iy} \) is large for \( y \rightarrow \infty \), whereas \( e^{-iy} \) is large for \( y \rightarrow -\infty \).

So, we could deal with each term separately!!

However, a nicer way to do this is to recall that for real \( x \), we have \( e^{ix} = \Re e^{ix} \).

Thus, \[ I = \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)} \, dx \]

Consider \( \Re I = \int_{\infty}^{10} \frac{e^{ix}}{(x^2 + 1)} \, dx \)

\[ = \int_{\infty}^{10} \frac{e^{-x}}{(x^2 + 1)} \, dx \]

\[ \approx 1.91 \leq R \]

\[ \Rightarrow \text{we expect that } A \text{ is small, but we need to show this.} \]

Note that \[ |e^{ix}| = e^{-y} \leq 1 \text{ for } y > 0 \]

\[ \frac{1}{2(x^2 + 1)} < \frac{1}{b^{x+1}} \text{ for } x > 1 \]

\( \Rightarrow |A| < \frac{1}{b^{x+1}} (\text{length of path}) = \frac{R}{b^{x+1}} \to 0 \text{ as } R \to \infty \)
So, \( I = \lim_{n \to \infty} \text{Im}(I_n) = \text{Re}\left[\frac{\pi}{2} \cdot \text{Re}(\omega)\right] \)

\[ \text{Re}(\omega) = \lim_{x \to \infty} (x+1) e^{ix} = \frac{e^{-i}}{x+1} = \frac{1}{x} \]

So, \( I = \frac{\pi}{e} \).

Some examples are even more subtle.

How about:

\[ I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} \, dx ? \]

Try some trick:

\[ I = \text{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} \, dx \]

\[ I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} \, dx \]

\[ I = \frac{\pi}{e} \cdot \frac{1}{2} \cdot \pi = \frac{\pi^2}{2e} \]

So, back to our problem:

\[ \mathcal{A} < \frac{\pi k^2}{(2k)^2} \]

So, \( \lim I_n(\mathcal{A}) = 0 \)

\[ \Rightarrow \quad I = \frac{\pi}{e} \]

Note: we also get for free

\[ \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} \, dx = \text{Re}\left[\frac{\pi}{2} \cdot \text{Re}(\omega)\right] = 0 \]

Though this was already clear by symmetry.
II. Jordan's Lemma

When does this trick work? Summarized by

\[
\lim_{\theta \to \infty} \int_{1/e^2}^{e^{2\theta}} e^{-iz} f(z) = 0
\]

If \( \Theta \to 0 \) \( f(z) \to 0 \) as \( z \to \infty \) for all \( \Theta > 0 \)

or \( \text{and} \quad f(z) \to 0 \) as \( z \to \infty \) for all \( \Theta > 0 \)

Our work above proves this result