Lecture # 13: Clever Contours

I. Using Symmetries

II. Integrals of multivalued functions

I. Using Symmetries

An amazing array of integrals can be evaluated by choosing clever contours, often one chooses a contour based on the symmetries of the problem.

For example, we did $I_3 = \int_{0}^{\infty} \frac{dx}{1 + x^m}$ by using the semi-circular contour.

That the integral is even we write this as

$I_{2n} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}} = \int_{0}^{\infty} \frac{dx}{1 + x^{2n}}$

but how would we evaluate

$I_1 = \int_{0}^{\infty} \frac{dx}{1 + x}$

The integrand is neither even nor odd.

However, it is invariant under $z \rightarrow e^{2\pi i/3}z$ so under

$I_1' = \int_{0}^{\infty} \frac{dx}{1 + x} = \frac{1}{e^{2\pi i/3}} \int_{0}^{\infty} \frac{dx}{1 + x}$

So

$I_1 = \frac{1}{e^{2\pi i/3}} I_1' = \frac{1}{e^{2\pi i/3}} \int_{0}^{\infty} \frac{dx}{1 + x}$

$I_3 = \frac{2\pi i}{2\pi i} \left( e^{2\pi i/3} \right) = 2\pi i \left( \frac{1}{1 + e^{2\pi i/3}} \right)$

So

$I_3 = \frac{2\pi i}{2\pi i} \left( e^{2\pi i/3} \right) = 2\pi i \left( \frac{1}{1 + e^{2\pi i/3}} \right)$

Note: Some trick works here for non-integer powers.

E.g., $I_p = \int_{0}^{\infty} \frac{dx}{1 + x^p}$ for $p > 1$ or convergence.

Note: $2\pi i$ is clockwise.

For $p > 0$

$\implies$ Branch cut is not a problem.

$I_\infty = \int_{0}^{\infty} \frac{dx}{1 + x^t}$ for $t > 1$ or convergence.

I'll let you try some other clever contours on your own next time.

I've prepared you though on one problem I wrote.

II. Integrals of multivalued functions

Sometimes we do need to think carefully about branch cuts, and sometimes they are even useful!

Consider $I_p = \int_{0}^{\infty} \frac{dx}{x^p}$ for $1 < p \leq 2$ for convergence.
We would like to make this into some useful contour integral. Recall, however, that $x^p$ is not single-valued in the complex plane. To make it single-valued, we must introduce a branch cut that runs from one to another curve.

For this problem it turns out to be useful to place the branch cut along the positive real axis right when we want to integrate!

We can now use the matching-ness to our advantage.

Consider:

$$I_p = \lim_{R \to \infty} \int_0^R \frac{d\zeta}{2\pi i (\zeta + i\epsilon)}$$

$$= \int_0^R \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

$$= \int_{-\infty}^{\infty} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)} + \int_{\infty}^{R} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

$$= \int_{-\infty}^{\infty} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)} - \int_{0}^{R} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

$$= \int_{-\infty}^{\infty} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)} - \int_{0}^{R} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

$$= \int_{-\infty}^{\infty} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)} - \int_{0}^{R} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

$$= \int_{-\infty}^{\infty} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)} - \int_{0}^{R} \frac{2\pi i \rho d\tau}{2\pi(\tau + i\epsilon)}$$

So

$$I_p = \frac{2\pi i \rho \log(\epsilon)}{2\pi i (1 - e^{-i\pi p})}$$

$$= \frac{2\pi i \rho \log(\epsilon)}{2\pi i (1 - e^{-i\pi p})}$$

$$= \frac{\pi - 2\pi}{e^{\pi i p} - e^{2\pi i p}} = \frac{\pi}{5\pi (1 + p)}$$