Lecture #15: More Conformal Maps

I. When does this work?

II. Examples

1. To a large extent, you can start from either end. i.e. Try to find \( g: D_2 \rightarrow D_2 \) analytic & take \( f = g^{-1} \).

Then, if \( z_0 = g(w_0) \) at a regular point of \( g \) with \( g'(w_0) \neq 0 \), then there is an inverse function \( f \) at least on a small disk around \( z_0 \) (i.e. \( f \) is analytic).

**Sketch of Proof:** Near \( w_0 \), \( g(w) = g(w_0) + g'(w_0) [w - w_0] + o(w - w_0) \).

\[
(z - z_0) = g'(w_0) [w - w_0] + o((w - w_0)^2)
\]
we can continue to invert the Taylor's series expansion to find

\[ w = f(z) \quad \text{and clearly} \quad f'(z_0) = \frac{1}{g'(z_0)} \]

since this where the series converges.

Note: Something interesting may happen if \( f' = 0 \) (\( \Rightarrow \) \( z_0 \) is singular) \( \text{or} \quad g' = 0 \) (\( \Rightarrow \) \( z_0 \) is singular).

\( 2 \) So long as \( z_0 \) is regular & \( f(z_0) \neq 0 \)

\( f \) preserves the angles at which curves intersect at \( z_0 \).

Note: we already saw some aspect of this,

For \( u + iv = z \), recall that the lines

Constant \( u \) & \( v \) are orthogonal in the \( xy \) plane.

\[ x + iy = f^{-1}(u + iv) \]

General proof: Consider two curves \( C_1, C_2 \) that intersect at \( z_0 \).

\[ w = w_0 + f'(z_0)(z - z_0) + o(z - z_0)^2 \]

Consider \( z \) near \( z_0 \),

Then \[ w = w_0 + f'(z_0)(z - z_0) + \text{small} \]

Let \( f'(z_0) = pe^{i\alpha} \)

Now, near \( z_0 \), each curve \( C_1, C_2 \) may be approximated by the straight (tangent) line through \( z_0 \). Any such line has an equation \( z - z_0 = re^{i\theta} \) for fixed \( r \).

Let \( r \) vary over \((-\infty, \infty)\)
So \( f(z) : w - w_0 = (p \nu) e^{i(\theta_1 + \omega)} + q(w - w_0) \)
\( f(z) : w - w_0 = (p \nu) e^{i(\theta_1 + \omega)} + q(w - w_0) \)

Both curves are rotated by the same angle \( \alpha \)

\( \Rightarrow \text{relative angle} \)

\( \theta_1 - \theta_2 \)

Any map that preserves local angles in this way is called \textbf{conformal},

i.e., \( f \) is a conformal map so long as

1) \( f \) is analytic & \( f' \neq 0 \).

\underline{Moral: If you want to introduce new corners or change the angle at which curves intersect, either \( f \) or \( f^{-1} \) must be singular.}

\underline{Quick Example}

Find the electric field outside a conductor bent at an angle \( \alpha \):

\[
\begin{align*}
\theta &= \alpha \\
\phi &= 
\end{align*}
\]

Map this to

\[
\begin{align*}
\theta &= 0 \\
\phi &= \pi \\
\end{align*}
\]

i.e., map \( \theta = \alpha \to \phi = \pi \)

Try \( w = \frac{z}{\pi/\alpha} \)

(analytic & single-valued for \( r > 0 \), \( 0 < \alpha < \pi \))

\[
\begin{align*}
\vec{E} &\text{ constant above infinite plane } \Rightarrow \phi = E_0 \hat{\nu} \\
\phi &= 0 \\
\phi &= \Re(-iE_0 z) \\
\phi &= f_2(z) = \Re(-iE_0 z^{\pi/\alpha})
\end{align*}
\]
\[
\phi = \text{Re} \left( e^{i E_0 r^{\pi/\omega}} e^{i \pi \frac{\theta}{\omega}} \right) \\
= E_0 r^{\pi/\omega} \text{Re} \left( -i \left[ \cos \left( \pi \frac{\theta}{\omega} \right) + i \sin \left( \pi \frac{\theta}{\omega} \right) \right] \right) \\
= E_0 r^{\pi/\omega} \sin \left( \pi \frac{\theta}{\omega} \right)
\]

Note: we can repeat this trick to introduce more corners.

\[ f^{-1}(t) = w^{4/\pi} \rightarrow \phi^{-1}(t) = (t - a_0)^{4/\pi} \]