Lecture #16: Yet More Conformal Maps

I. Boundary Conditions & Conformal Maps

II. Some Cool Examples

I. Boundary Conditions & Conformal Maps

Last time we noticed that we could use conformal maps (i.e., analytic functions) to solve Laplace's equation by mapping the complicated BC's to simple ones.

\[ f \] \[ f_1 \]

\[ B_w \]

\[ B_z \]

We arranged \( f \) to map one boundary to another and saw that \( \text{Re } f_2(z) \)
satisfied the desired boundary conditions (BCs).

**Question:** Is this always the case?

Let's look at the most common BCs:

**Dirichlet:** Fix \( \phi = \text{constant on the boundary} \)

Simple sol: \( \phi = \text{const} \) for \( w \in B_w \)

New sol: \( \phi = \text{Re } f_2(C) \)

For \( z \in B_2, \ w = f(z) \in B_w \)

\[ \Rightarrow \phi = \phi_0 \quad \checkmark \]

**Neumann:** Fix \( \hat{n} \cdot \nabla \phi = 0 \) on the boundary

Simple sol: \( \hat{n}_w \cdot \nabla \phi = 0 \) on \( B_w \)

\( \Rightarrow \) For any curve \( C \subset B_w \) and any parameter along that curve, \( \frac{d\phi}{ds} \bigg|_{B_w} = 0 \).
Complicated Sol: \[ \text{want } \int_0^\pi \dot{\psi} \cdot n dt = 0 \text{ on } B^2 \]

to test this, consider some curve \( C^2 \subset B^2 \)

& some parameter \( \lambda \) along this curve.

\[
\frac{\partial \phi}{\partial \lambda^2} = 7.
\]

Note: \( f \) maps \( C^2 \) to some curve \( C_w \).
Suppose \( C^2 \) intersects \( B^2 \) at a regular pt \( z_0 \) of \( C \).
Then \( f \) is conformal at \( z_0 \) & \( C_w \perp B_w \)
at \( w_0 = f(z_0) \),

Any parameter \( \lambda \) along \( C_w \) is some function \( \lambda = (\lambda(\theta)) \).

Along \( C^2 \) we have \( \phi = \Re f_2 (z) \).
\[
\Rightarrow \frac{\partial \phi}{\partial \lambda^2} = \frac{2\pi}{2\pi} \frac{2\lambda}{2\pi} = 0
\]

So, the Neumann condition is preserved at every regular point of \( f \).

II. Some cool Examples

To get an idea of what is possible, let's explore some simple conformal maps. Then, as noted last time, we can assemble several of these together to do something complicated.

1. Last time: use \( w = z^{\pi/\alpha} \) to make a corner with angle \( \alpha > 0 \)

\[
\frac{z}{\text{Real axis}}
\]
Another simple conformal map is:

\[ w = \frac{1}{z} \]

But what does it do to straight lines?

No longer straight...

\[ z = \frac{1}{uv} \]

Might this be a circle? Centre would be at \( z_0 = \frac{1}{v} \)

Example:

\[ z - z_0 = \frac{1}{uv} - \frac{1}{2v} = \frac{2iv - u - iv}{2iv(u + iv)} = \frac{iv - u}{2iv(u + iv)} \]

\[ = \frac{-1}{2iv} \frac{v - iv}{u + iv} \]

\[ \Rightarrow |z - z_0| = \frac{1}{\sqrt{v^2 - 1}} \text{ constant!} \]

So indeed our straight line \( v = \text{const} \) maps to a circle.

Note: All other straight lines are related to one \( v = \text{const} \) by a rotation \( \Rightarrow \) All straight lines map to circles.

\[ \text{One may also show that any circle maps either to a straight line or another circle.} \]

Clearly the same is true of:

\[ w - w_0 = \frac{z - z_0}{\beta(z - z_0) + \alpha} \quad \text{or} \quad w = w_0 + \frac{\alpha}{\beta(z - z_0)} = \frac{\beta w_0 z + (\alpha - \beta w_0 z_0)}{\beta(z - z_0)} \]

\[ \beta(z - z_0) \]

Some parameters are redundant:

\[ \begin{align*}
q &= \omega_0, & b &= \alpha - \omega_0 \beta z_0, & c &= \beta, & d &= \beta z_0
\end{align*} \]

Rename the parameters:

\[ w = \frac{az + b}{cz + d} \]

The standard form for fractional linear transformations.
Our work above shows that such maps transform straight lines into circles.

Let's use this in an example.

Potential in a cylindrical space

Electrostatics

Find the potential inside.

Let's try to map this to a simpler problem. We can use a fractional linear transformation to map this circle to a straight line.

Let's try \( w = x \frac{z+i}{z+i} \)

Try to send our circle to the real axis, perhaps

\( z = 1 \rightarrow w = x \Rightarrow a = -1 \)

\( z = -1 \rightarrow w = x \Rightarrow d = 1 \)

For \( z = e^{i\theta} \)

\[ w = \alpha \frac{z - i}{z + i} = \alpha \frac{e^{i\theta} - i}{e^{i\theta} + i} = \alpha \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2}} \]

\[ = \alpha \frac{2i \sin \theta}{2 \cos \theta} = i \alpha \tan \theta \]

To map this circle to the real axis, choose \( \alpha = -i \)

So that \( e^{i\theta} \rightarrow w = \tan \theta \in \mathbb{R} \)

\[ w = -i \frac{z - i}{z + i} \text{ Great!} \]

So, in \( w \)-plane our problem is

\( \phi = 1 \) \( \phi = 0 \)

Insulation

But this is just like the example we solved last time! Try \( \log w \)
For \( w = re^{i\beta} \), \( \log w = \log r + i\beta \)

So, let \( \phi = \frac{1}{\pi} \text{Im} \log w = \frac{1}{\pi} \tan^{-1}(\frac{y}{x}) \)

\[ \Rightarrow \phi = \frac{1}{\pi} \tan^{-1}\left[ \frac{y}{\sqrt{x^2+y^2}} \right] \]

\[ u = \frac{2y}{(x^2+y^2)} \quad \Rightarrow \quad v = \frac{1-x^2-y^2}{(x^2+y^2)} \]

\[ \Rightarrow \phi = \frac{1}{\pi} \tan^{-1}\left( \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{1}{\pi} \tan^{-1}\left( \frac{1-x^2-y^2}{2y} \right) \]

**What does this look like?**

**The equipotential curves are**

\( \psi = \text{constant} = \frac{1-x^2-y^2}{2y} \), or

\[ 1 = x^2 + 2cy + y^2 \]

\[ = x^2 + (y + c)^2 - c^2 \Rightarrow x^2 + (y + c)^2 = 1 + c^2 \]

These curves are all circles & all pass through the two insulators at \( x = \pm 1, y = 0 \).

\[ \Rightarrow \text{Sol } \Psi \]

**The Electric Field is the gradient, \( \mathbf{E} \) to the \( \phi \)-constant lines.**

Cool, eh? Clearly we can put such transformations together to study lots of interesting geometries.