I. Some Final Applications of Conformal Transformations

Before the exam we were discussing the use of conformal maps to solve the Laplace Equation with complicated boundary conditions. We'll spend today doing one more example (of particular interest!) to get you warmed up again and ready to attack the new HW. (There you will get ample opportunity to explore other interesting applications.) This will finish the part of the course that is specifically about complex analysis, though we will continue to use the tools we've built for the rest of the course. Next week we will discuss Fourier Transforms which I understand you skipped this year in 100 A.

Our final example involves boundaries that end.

Consider the interesting conformal map

\[ z = w + \frac{1}{2} e^{\pi i w} \]  

\[ x = u + \frac{1}{2} e^{\pi i} \cos(\pi v) \]  

\[ y = v + \frac{1}{2} e^{\pi i} \sin(\pi v) \]

Note that we have written this "backwards" on two!

That \( w = w(z) \) is hard to find. Nevertheless, this map will be useful. I'm not sure what first prompted someone to try this map, but it does some useful things.
Answer: Compute $\frac{d^2}{dw} \phi(w)$ is entire, so $\phi(w)$ exists. For $\frac{d}{dw}$, we will exist & be analytic.

Sometimes, we

$$\phi = \frac{z^2}{\phi} \left( \frac{1}{\phi} \right)$$

Now! We just solved this equation:

$$u = \frac{d}{dw} \phi, \quad v = \frac{d}{dw} \phi \cdot \ln \phi$$

That's why the plates "break" at $\phi = 0$.

What about the other singularities? Let's return to this question later.

OK, what does our solution look like?

If $\phi = \text{Im} \phi = v$

$$v = \frac{4\pi}{d} e^{\frac{2\pi}{d} z} \sin \left( \frac{2\pi}{d} \phi \right)$$

$$\phi = \frac{1}{2} \frac{d}{dw} \sin \left( \frac{2\pi}{d} \phi \right)$$

Let's try to plot equipotential surfaces.

For $v = \phi$, $u$ is not a very.

$$v = \frac{1}{2} \frac{d}{dw} \sin \left( \frac{2\pi}{d} \phi \right)$$

Note: We cover the entire $z$-plane

The values $\frac{d}{2} \geq d \geq -\frac{d}{2}$

That means that we used only $\phi = \text{Im} \phi \geq 0$.

This is in the $w$-plane maps to the entire $z$-plane.

Ah! $w = w(z)$ is multi-valued.

we must fix this by putting branch-cuts along the plates (where we don't have to satisfy $\phi = 0$ because we imposed a boundary condition).

These branch cuts stop us from ever reaching a region containing the other singularities! [So we can forget about them!]

our solution does not go to zero $\phi$-plane away from the plates!}
We can also draw the electric field lines.

They are ± to the equipotentials:

Recall that we can also interpret our diagrams in terms of fluid flow. In that case the \( \phi \) constant lines become streamlines.

In this center our solution describes a pert emptying into a large lake in the case of zero viscosity.