Lecture 21: Solving ODE's with Fourier Transforms

I. The Damped Harmonic Oscillator

II. Restitution Again?

III. Response to a Finite Time Driving Force

IV. An Explicit Solution

I. The Damped Harmonic Oscillator

We spent a week discussing Fourier Transforms. Let's use them to do some physics!!!

You are all familiar with harmonic oscillators.

\[ F = -kx \]
\[ m \dot{x} + \omega^2 x = 0 \]

You have also seen damped oscillators, where one adds a velocity-dependent friction force:

\[ F_{\text{damp}} = -b \dot{x} \]
\[ m \ddot{x} + 2b \dot{x} + \omega^2 x = 0 \]

You may know how to solve that equation... but can you do so in the presence of an extra external applied force \( F(t) \)?

\[ x + \omega^2 t + 2b \dot{x} = \frac{1}{\Lambda(t)} F(t) \]

This is easy to do using Fourier Transforms! We assume \( \Lambda(t) \rightarrow 0 \) as \( t \rightarrow \infty \). So our Fourier Transforms exist.

We also assume that \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Thus, that its Fourier Transform exists.

Note that if \( F(t) = 0 \), this would mean that

\[ x(t) = \frac{1}{\Lambda(t)} \int_{-\infty}^{\infty} e^{-i\omega t} F(\omega) d\omega \]

Note that in principle, we have solved the problem: we have expressed \( x(t) \) in terms of \( \Lambda = \frac{b}{m} \).

\[ X(\omega) = \frac{1}{\Lambda(t)} \int_{-\infty}^{\infty} e^{-i\omega t} F(\omega) d\omega \]

\[ x(t) = \frac{1}{\Lambda(t)} \int_{-\infty}^{\infty} e^{-i\omega t} \Lambda(t) d\omega \]

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But, wouldn't it be nice to have a way to do this integral?

Do we get an explicit solution?

II. Residues again!

we can do so for some interesting cases by using contour integration!!!

Let's give this a clever way.

\( \Phi(z) = \Phi_0(z) \)

The proper

transformation of this part is just \( \Phi_0(x) \).

If we use contour integration, it is also not hard to calculate

\[
c(t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-t \omega^2} \omega \, d\omega
\]

Simple poles at \( \omega = \pm \sqrt{i \omega_0} \).

\[
\omega = \frac{-\sqrt{\omega_0 \pm \sqrt{\omega_0^2 + 4 \alpha \beta}}}{2}
\]

Three cases:

1. "underdamped" \( \omega^2 < \omega_0^2 \)

\[
\Rightarrow \omega = \pm \sqrt{\omega_0^2 - \alpha^2}
\]

poles only in the upper half plane (for \( \omega > 0 \))

2. "critically damped" \( \omega = \omega_0 \)

\( \omega = \omega_0 \) in upper half-plane

3. "overdamped" \( \omega^2 > \omega_0^2 \)

\[
\omega = \pm \sqrt{\omega_0^2 \pm \alpha^2}
\]

\[
\Rightarrow \omega = \omega_0, \omega_0 > 0
\]

Again, poles only in the lower half-plane.

\[
G(s) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-t \omega^2} \omega \, d\omega
\]

For \( s \leq 0 \) close curves in upper half plane

\[
\text{circular} \quad e^{\lambda t} \text{or} \quad e^{-\lambda t}
\]

\[
\Rightarrow G(s) = 0 \quad \text{for} \quad s \leq 0
\]

why would that be? note: \( \Phi(x) \) is the response

when \( \Phi_0(x) = 1 \), what kind of driving force is that?

note: \( \Phi_0(x) \neq \Phi \)?

Answer: a Dirac distribution!

\[
\text{Proof:} \quad \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-t \omega^2} \Phi(x) \, d\omega = \frac{e^{t x^2}}{\sqrt{\pi t}} = \frac{1}{\sqrt{\pi t}}
\]

\[
\text{so,} \quad \Phi_0(x) = 1 \Rightarrow \Phi = \frac{x}{\sqrt{\pi t}} \Phi_0(x)
\]

This would represent an impulsive driving force

at \( x = 0 \). Thus the response before \( x = 0 \)!

Finally compute \( B(t) \) for \( \lambda > 0 \).

from most close center in lower half-plane

\[
\Rightarrow \text{get contributions from poles,}
\]

\[
G(s) = \frac{e^{t x^2}}{\sqrt{\pi t}} \left( \sum \text{of Residues} \right)
\]

\[
\frac{e^{t x^2}}{\sqrt{\pi t}} \left( \sum \text{of Residues} \right)
\]

\[
\frac{e^{t x^2}}{\sqrt{\pi t}} \left( \sum \text{of Residues} \right)
\]

where \( \omega = -i \omega_0 \pm \sqrt{\omega_0^2 \pm \alpha^2} \).
Suppose simple poles \((m_i, x_m)\) are critically damped.

\[
\mathcal{R}_m(s) = -\frac{e^{-sx_m}}{(s^2 + w^2)} = -\frac{e^{-sx_m}}{2\sqrt{w^2 - x^2}}
\]

\[
\mathcal{R}_m(s) = -\frac{e^{-sx_m}}{s} = \frac{e^{-sx_m}}{2\sqrt{w^2 - x^2}}
\]

So, sum of residues:

\[
\frac{e^{-sx_m}}{2\sqrt{w^2 - x^2}} - \frac{e^{-sx_m}}{2\sqrt{w^2 - x^2}} = \frac{2}{2\sqrt{w^2 - x^2}}
\]

Case 1) Undamped, \(w^2 - x^2 > 0\) let \(s = \sqrt{w^2 - x^2}\)

\[
\mathcal{G}(s) = \frac{\sqrt{w^2}}{2\pi} \frac{e^{-sx_m}}{s} (\cosh(\sqrt{w^2 - x^2}) - e^{-sx_m})
\]

\[
= \frac{\sqrt{w^2}}{2\pi} \frac{e^{-sx_m}}{s} \sinh(\sqrt{w^2 - x^2})
\]

---

Shall we do the critically damped case?

\[
\mathcal{G}(s) = \frac{1}{\sqrt{w^2}} \int \frac{e^{-sx_m}}{w^2 - x^2} \, dx = -\frac{1}{\sqrt{w^2}} \int \frac{e^{-sx_m}}{w^2 + \alpha^2} \, dx
\]

\[
\text{extra } (-15\pi^2)
\]

For retarded \(\alpha > 0\)

\[
= \frac{2\pi e^{-sx_m}}{w^2} \mathcal{R}_m(-\alpha)
\]

Prove pole \(\alpha \rightarrow \mathcal{R}_m = \frac{1}{\alpha} e^{-sx_m} \mid_{\alpha = -\alpha}
\]

\[
= \frac{e^{-sx_m}}{-\alpha}
\]

\[
\mathcal{G}(s) = \frac{\sqrt{w^2}}{2\pi} \frac{e^{-sx_m}}{s} \sinh(\sqrt{w^2 - x^2})
\]

I \(\text{ damping as fast as possible}
\]

"Critically damped"

---

III. Response to a Finite Time Driving Force

So, we were able to find \(\mathcal{G}(s)\) = response to an impulse at \(t = 0\).

What happens in general?

\[
\mathcal{X}(s) = \mathcal{A}(s) \mathcal{G}(s)
\]

\[
\mathcal{X}(s) = \frac{1}{\sqrt{w^2}} \int \frac{e^{-sx_m}}{w^2 - x^2} \mathcal{A}(s) \mathcal{G}(s)
\]

\[
\mathcal{X}(s) = \frac{1}{\sqrt{w^2}} \int \mathcal{A}(s) e^{-sx_m}
\]

\[
\mathcal{X}(s) = \frac{1}{\sqrt{w^2}} \int \mathcal{G}(s) e^{-sx_m}
\]
\[ X(\tau) = \frac{1}{(2\pi i)^N} \int_{\Gamma_1} e^{2\pi i \tau \xi} \int_{\Gamma_2} A(z) e^{2\pi i z \xi} \int_{\Gamma_3} G(z \xi) e^{2\pi i z \xi} \]

**Trick:**
Recall that
\[ G(\tau \cdot \xi) = \langle \xi, \tau \rangle G(\xi) = \frac{1}{(2\pi i)^N} \int \ldots e^{-2\pi i \tau \cdot \xi} e^{2\pi i \xi \cdot \xi} \]

So \[ \int \ldots e^{2\pi i \tau \cdot \xi} = 2\pi i \delta(\tau \cdot \xi) \]

So, do \[ w \cdot \text{integral} \text{ First} \]

\[ X(\tau) = \frac{1}{(2\pi i)^N} \int \ldots A(z) G(z \cdot \tau \cdot z) \]

Recall that the defining property of a delta function is that
\[ \int \ldots e^{2\pi i \tau \cdot \xi} \delta(\tau \cdot \xi) = f(\tau) \]

So, to perform the integral over \( \tau \) we merely set \( \tau_1, \ldots, \tau_N = 0 \)

i.e. \[ \tau \cdot \xi = \tau \cdot \xi \]

\[ \Rightarrow X(\tau) = \frac{1}{(2\pi i)^N} \int \ldots A(z) G(z \cdot \tau \cdot z) \]

This is called a "convolution"
\[ \equiv A \ast G \]

i.e. \[ X(\tau) = \frac{1}{(2\pi i)^N} (A \ast G)(\tau) \]

**Note:** We learn that the (inverse) Fourier transform of a product is the convolution of the Fourier transforms. Some is the Fourier transform.

Also note: This formula makes sense.

\[ G(\tau) \text{ is the response to } A = \frac{1}{2\pi i} \int_0^\infty S(\xi) e^{-2\pi i \xi \cdot \tau} d\xi. \]

What is response to \( A(\tau) = S(\tau - \tau) \)?

Note: Equation is linear & invariant under shifts of \( \tau \).

**Answer:** \[ X(\tau) = \frac{1}{2\pi i} \int_0^\infty G(\tau - \tau) \]

So, what is the response to a general

\[ A(\tau) = \int \ldots A(z) G(z \cdot \tau \cdot z) \]

**Answer:** \[ X(\tau) = \frac{1}{2\pi i} \int_0^\infty \int \ldots A(z) G(z \cdot \tau \cdot z) \]

This \[ G(z \cdot \tau \cdot z) \] is called a "Green's Function!"

We could have used the idea of a Green's function to solve the equation from the start. Often it is not hard to solve equations of delta-function sources.

We may come back to Green's functions later if we have time.

Now, if I give you a simple \( A(\tau) \) (e.g., \( \frac{1}{\tau^2} \)) you can do this last integral \( (\ldots) \) to compute \( X(\tau) \).

Or, in some cases it might be easier to look at (as below)

\[ X(\tau) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-2\pi i \tau \cdot \xi} A(\xi)}{e^{2\pi i \xi \cdot \tau} - e^{2\pi i \tau \cdot \xi}} \]

\[ (\ast) \]
II. An Explicit Solution

Let's work out the final answer for, e.g., the critically-damped case. I'll leave the other cases for you to do on your own.

\[ A(x) = \sqrt{\frac{\Delta A_0}{2}} \]

\[ x(t) = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{\infty} A_0 e^{-\frac{(x-x_1)^2}{4c}} \, dx_1 \]

For \( x_1 > T \)

\[ x(t) = A_0 e^{-\alpha t} \int_0^T dx_1 e^{\alpha (x-x_1)} \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} - \frac{e^{\alpha T}}{\alpha} \int_0^T dx_1 e^{\alpha (x-x_1)} \right] \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} - \frac{e^{\alpha T}}{\alpha} \left( \frac{e^{\alpha T}}{\alpha} \right) \right] \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} - \frac{e^{\alpha T}}{\alpha} \left( \frac{e^{\alpha T} - 1}{\alpha^2} \right) \right] + \alpha T \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} + \frac{e^{\alpha T} - 1}{\alpha^2} \right] \]

\[ = A_0 \left[ -\frac{\alpha}{c} e^{-\alpha t} + \frac{1}{\alpha^2} (1 - e^{-\alpha t}) \right] \text{ for } T > \frac{x}{\alpha} \]

Ah! we did it!

Note that it has the expected properties:

1. damped \( \alpha e^{-\alpha t} \)

2. Biper response if time applied for longer time grows like \( e^{\alpha T} \)

3. \( \rightarrow 0 \) as \( T \rightarrow 0 \)

4. Since \( T \) \( \rightarrow \), \( \rightarrow 0 \) as \( \rightarrow \infty \).

Oh, we should still do the case \( x < T \).

Note: \( x < 0 \) is easy: \( x(t) = 0 \)

For \( T > x \geq 0 \) we have \( e^{\alpha (x-x_1)} = 0 \) for \( x < x_1 \).

\[ x(t) = A_0 e^{-\alpha t} \int_0^x dx_1 e^{\alpha (x-x_1)} \text{ for } x < x_1 \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} - \frac{e^{\alpha T}}{\alpha} \left( \frac{e^{\alpha T} - 1}{\alpha^2} \right) \right] + \alpha T \]

\[ = A_0 e^{-\alpha t} \left[ \frac{e^{\alpha T} (x-x_1)}{\alpha} + \frac{e^{\alpha T} - 1}{\alpha^2} \right] \]

\[ = A_0 \left[ -\frac{\alpha}{c} e^{-\alpha t} + \frac{1}{\alpha^2} (1 - e^{-\alpha t}) \right] \text{ for } T > \frac{x}{\alpha} \]