Lecture #22: Laplace Transforms

I. When Fourier Transforms Fail
II. Properties of Laplace Transforms
III. Examples

I. When Fourier Transforms Fail

We have seen that Fourier transforms are useful tools for solving differential equations. However, we defined Fourier transforms by thinking about the operator \( \mathcal{L}(f) = \int_{-\infty}^{\infty} e^{-ixt} f(x) \, dx \). So what do we do if the relevant functions are not in \( L^1(\mathbb{R}) \)?

E.g., \( f(x) = \text{constant} \) or \( f(x) = x^2 \)?

If \( f(x) = e^{-cx} \) then we can use the Laplace Transform. The essence is to transform the function \( f(x) e^{-cx} \), which transforms to have nice properties. However, \( f(x) = e^{-cx} \) is typically badly behaved as \( x \to -\infty \). The Laplace Transform is used when we only care about the behavior for \( x > 0 \) (or more generally \( x > a \)) so that we can simply remove the part \( f(x) \) for \( x < a \).

Specifically, consider the so-called "Heaviside step function" \( \Theta(x) \) defined as:

\[
\Theta(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0 
\end{cases}
\]

[Note: \( \Theta(x) \) is \( L^1(\mathbb{R}) \) since \( \int_{-\infty}^{\infty} \Theta(x) \, dx = 1 \).]

The function \( \Theta(x) f(x) e^{-cx} \) lies in \( L^1(\mathbb{R}) \) and can be Fourier Transformed.

\[
\mathcal{L}(f(x) e^{-cx}) = \int_{-\infty}^{\infty} f(x) e^{-cx} e^{-ixt} \, dx
\]

By the convolution theorem, it is easy to find the inverse transform

\[
\mathcal{L}^{-1} \left[ \frac{1}{s} \right] = e^{-CX}
\]

The convolution theorem states

\[
\mathcal{L}(f(x) e^{-cx} e^{-ixt}) = \int_{-\infty}^{\infty} f(x) e^{-cx} e^{-itx} \, dx
\]

The inverse transform is

\[
\Theta(x) f(x) e^{-cx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \mathcal{F}(f(k)) \, dk
\]

using our rules for Fourier Transforms.

So generally, there will be some minimum value \( C \) where \( \Theta(x) \) starts.

By \( \text{Cayley} \)

The Laplace transform just involves redefining variables:

Let \( s = \sigma + i \omega \)

We have

\[
\mathcal{L}(f(x)) = \int_{0}^{\infty} f(x) e^{-xt} \, dx
\]

Now consider \( f(x) e^{-cx} \) and

\[
f(x) e^{-cx} = \int_{0}^{\infty} \mathcal{F}(f(x)) e^{\omega x} \, d\omega
\]

The integral (4) generally converges only in the

region \( c - \omega > 0 \), since \( \mathcal{F}(f(x)) \) is analytic.

Thus \( f(x) e^{-cx} \) converges for \( \text{Re} \{s\} > C \).

However, we may be able to also define \( \mathcal{L}(f(x)) \) to the left of \( C \) by "analytic continuation." This simply means that we
may be able to find an analytic function that makes sense for $R(s) > 0$. But that agrees with $F(t)$ for $R(s) > 0$. [Such an analytic continuation is typically unique up to a choice of branch cuts; we'll discuss this later.]

**Inverse Transform**

$$f(x) = \frac{1}{2\pi i} \int_{C_+} F(s) e^{sx} ds.$$ We typically compute this by closing the contour.

Note that $F(s) = \int_0^\infty f(x) e^{-sx} dx \to 0$ as $s \to \infty$.

For $x > 0$ we get the $s$-limit since we can close the contour in the right half-plane where $F(s)$ is analytic.

For $x < 0$ we will try to close the contour on the left and use the residue theorem to evaluate the integral.

**Properties of Laplace Transforms**

Laplace transforms are typically used in initial value problems where, even if we have to solve for $f(t)$, we already know $f(t)$ for $t < 0$. This is useful to note.

1. **Laplace Transform of a Derivative**

$$L \left\{ f'' \right\} = L \left\{ \frac{d^2}{dt^2} f(t) \right\} = \int_0^{\infty} f''(x) e^{-sx} dx$$

2. **2nd Derivative**

$$L \left\{ x^2 \right\} = L \left\{ \frac{d}{dt} f(t) \right\} = -f'(x) + sL[f(t)]$$

3. **Laplace Transform of a Product**

$$L \left\{ x f(x) \right\} = \int_0^{\infty} x f(x) e^{-sx} dx$$

4. **Laplace Transform of a Sum**

$$L \left\{ f(x) + g(x) \right\} = L \left\{ f(x) \right\} + L \left\{ g(x) \right\}$$

5. **Convolution**

$$L \left\{ f * g \right\} = \int_0^{\infty} f(x) g(x-y) dy$$

Then $L \left\{ f * g \right\} = L \left\{ f \right\} L \left\{ g \right\}$

**Proof:** Just like for Fourier transforms.

**III. Examples**

A. Solve $y''(t) + 4y(t) = \cos(3t)$

$y(0) = 2$, $y'(0) = 0$. 

Let's try to Laplace transform this equation

\[ \mathcal{L}\{y(t)\} = \mathcal{L}\{G(t)\} \]

\[ \mathcal{L}\{y(t)\} = s\mathcal{L}\{y(t)\} - y(0) - y'(0) = s\mathcal{L}\{y(t)\} - 2s \]

\[ \mathcal{L}\{y(t)\} = \int_{0}^{\infty} e^{-st} y(t) \, dt = \frac{1}{2} \int_{0}^{\infty} e^{-st} (e^{-t} + e^{-at}) \, dt \]

\[ = \frac{1}{2} \left[ \frac{1}{s-t} + \frac{1}{s+a} \right] = \frac{s}{(s-a)(s+t)} \]

So, the Laplace transformed equation is

\[ \mathcal{L}\{F(\xi)\} - 2s + 2s = \frac{s}{(s-a)(s+t)} \]

\[ (s^2 + 2s + 9)F(\xi) = 2s + \frac{s}{(s-a)(s+t)} \]

\[ F(\xi) = \frac{\frac{s}{(s-a)(s+t)}}{(s^2 + 2s + 9)} + \frac{s}{(s-a)(s+t)} \]

To solve for \( y(t) \) we invert the transform:

\[ \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + \alpha} \right\} = \frac{1}{\sqrt{\alpha}} e^{-\frac{\alpha}{2}t} \]

\[ \text{For large } \alpha \text{, the term } e^{-\frac{\alpha}{2}t} \text{ will dominate.} \]

\[ \text{In summary, we have } y(t) = \text{some terms} \]

For \( \alpha > 0 \), close the contour on the right & set \( \alpha = 0 \).

For \( \alpha < 0 \), close the contour on the left & set \( \alpha = 0 \).

For \( \alpha = 0 \), close the contour on the left & set \( \alpha = 0 \).

For \( \alpha > 0 \), close the contour on the right & set \( \alpha = 0 \).

For \( \alpha < 0 \), close the contour on the left & set \( \alpha = 0 \).

By solving the differential equation, we find the solution:

\[ y(t) = 2\cos(\xi t) + \frac{1}{\xi^2} \left( \frac{1}{2} - e^{-\xi^2 t} \right) \]

The residue at \( \xi = -i \) is the complex conjugate:

\[ y(t) = e^{\xi t} \left( \frac{1}{\xi^2} \left( \frac{1}{2} - e^{-\xi^2 t} \right) \right) \]

We can also use Laplace transforms to solve initial value problems.

Suppose we know that

\[ f(x) = 1 + \int_{0}^{x} x e^{-\tau} f(x-x) \, d\tau \]

where \( f(x) = 0 \) for \( x < 0 \).
more that this implies that \( f(x-x') = 0 \) for \( x' \neq x \), so that the above equation can be written

\[
f(x) = 1 + \sum_{n=0}^{\infty} x^n e^{-sx^n} f(x-x^n) dx^n\]

Fig. 3 The convolution that acts nicely with respect to Laplace transforms

Let \( L[f] = F(s) \),

\[F(s) = L[I] + F(s) L[e^{-sx}]
\]

\[L[I] = \int_0^\infty \frac{1}{s} e^{-sx} dx = \frac{1}{s}
\]

\[L[e^{-sx}] = \int_0^\infty x e^{-sx} e^{-sx} dx = -\frac{1}{s} \int_0^\infty e^{-sx} e^{-sx} dx = -\frac{1}{s^2}
\]

\[= \frac{1}{s^2} \left( \frac{1}{s} \right)
\]

\[S_x^2 \quad F(s) = \frac{1}{s} + F(s) \left( \frac{1}{s^2} \right)
\]

\[S_x^{(5)} \quad F(s) = \frac{(s+1)^2}{s} + F(s)
\]

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\]

\[F(s) = \frac{(s+1)^2}{s^2} \quad \text{poles at } s = -1, 0, 2
\]

Now we do the inverse transform:

\[f(x) = \frac{1}{2\pi i} \int_C e^{sx} F(s) ds \quad \text{for } c > 0
\]

For \( M > 0 \), closing the contour on the left yields

\[f(x) = \text{sum of residues}
\]

\[= \mathcal{R}_{M_2}(s) + \mathcal{R}_{M_2}(-s)
\]

\[\mathcal{R}_{M_2}(-s) = \frac{e^{sM}}{s^2} \left[ \frac{e^{sM}}{s^2} - 1 \right]
\]

\[\mathcal{R}_{M_2}(s) = \frac{1}{2} e^{-sM}
\]

\[
E(s) e^{sx} = \frac{1}{2} \left[ 1 + (eM)^s + \cdots \right]
\]

\[= \frac{1}{2} \left[ 1 + (eM)^s + \cdots \right]
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\]

\[\Rightarrow \mathcal{R}_{M_2}(s) = \frac{1}{2} + \frac{1}{2}
\]

So \( f(x) = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} e^{-2x} = \frac{1}{4} \left[ e^{-2x} + 2x + \frac{3}{4} \right] \)