Lecture #22: Laplace Transforms

I. When Fourier Transforms Fail

II. Properties of Laplace Transforms

III. Examples

I. When Fourier Transforms Fail

We have seen that Fourier transforms are useful tools for solving differential equations. However, we defined Fourier transforms by thinking about the operator $dx$ on $L^2(\mathbb{R})$. So, what do we do if the relevant functions are not in $L^2(\mathbb{R})$?

E.g., $f(x) = \text{constant}$ or $f(x) = x^2$?

If $f(x) e^{-cx} \to 0$ as $x \to \pm \infty$ for some real $c > 0$, then we can use the Laplace transform. The essential idea is Fourier transform the function $f(x) e^{-cx}$, which turns out to have nice properties. However, $f(x) e^{-cx}$ is typically badly behaved as $x \to \pm \infty$. The Laplace transform is used when we only care about the behavior for $x > 0$ (or, more generally, $x > x_0$) so that we can simply ignore the part of $f(x)$ for $x < 0$.

Specifically, consider the so-called "Heaviside step function"

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

[note: \( \frac{d\Theta}{dx} = \delta(x) \), since \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \)]

The function \( \Theta(x) f(x) e^{-cx} \) lies in $L^2(\mathbb{R})$ and can be Fourier transformed.
Compute

\[ f(x) = \int_{-\infty}^{\infty} g(t) e^{-cx} e^{-itx} \, dt \]

[Here the convention is to put the \( \frac{1}{2\pi} \) in the inverse transform]

\[ = \int_{0}^{\infty} f(x) e^{-cx} e^{-ikx} \, dx \]

So the inverse transform is

\[ \Theta(x) f(x) e^{-cx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} F(s) \, ds \]

using our rules for Fourier transforms.

In general, there will be some minimum value of \( c \) where this works, (say, \( c = \infty \))

The Laplace transform just involves renaming variables:

Let \( S = c + ik \)

\[ F(s) = \mathcal{L}[f(x)](s) \]

We have

\[ F(s) = \int_{0}^{\infty} f(x) e^{-sx} \, dx \quad \text{and} \]

\[ f(x) \Theta(x) = \frac{1}{2\pi i} \int_{C_c} F(s) e^{sx} \, ds \quad \text{where } C_c \text{ is the} \]

\[ \text{contour } \Re S = C \text{ for any } C > \infty \]

The integral (*) generally converges only in the right-half \( s \)-plane \( C > 0 \). In this region, \( F(s) \) is analytic since

\[ F^{-1}(s) = \int_{0}^{\infty} f(x) (-s) e^{-sx} \, dx \]

converges for the same values of \( s \) that (*) does.

However, we may be able to also define \( F(s) \) to the left of this line by "analytic continuation." This simply means that we
may be able to find an analytic function that makes sense for $\text{Re}(s) < \alpha$ but that agrees with $F(s)$ for $\text{Re}(s) > \alpha$. [Such an analytic continuation is typically unique up to a choice of branch cuts; we'll discuss this idea more later.]

**Inverse Transform**

$$f(x) = \frac{1}{2\pi i} \oint_C F(s) e^{sx} \, ds$$

We typically compute this by closing the contour.

Note that $F(s) = \int_0^\infty f(x) e^{-sx} \, dx \to 0$ as $\text{Re} s \to \infty$.

Thus, for $x < 0$ we get $f(x) = 0$ since we can close the contour in the right half-plane where $F(s)$ is analytic.

For $x < 0$ we will try to close the contour on the left and to use the Residue Theorem to evaluate the integral.

**II. Properties of Laplace Transforms**

Laplace Transforms are typically used in initial value problems where, even if we have to solve for $f(x)$, we already know $f(0), f'(0), \ldots$. Thus it is useful to note the following:

1. **Laplace Transform of a Derivative**

$$L\left[ \frac{df}{dx} \right] = L\left[ f' \right] = \int_0^\infty f'(x) e^{-sx} \, dx$$

A useful notation for the Laplace transform is $f(s) = f(x) e^{-sx} \mid_{x=0} + s \int_0^\infty f(x) e^{-sx} \, dx$.

So $f(x) = 0 + s L[f] - f(0)$.
2nd Derivative

Use the above rule twice

\[ L \left[ f'' \right] = L \left[ \frac{d f'}{d x} \right] = -f'(a) + s L \left[ f' \right] \]
\[ = -f'(a) - sf(a) + s^2 L \left[ f \right] \]

3. \[ L \left[ e^{xt} \right] \]

\[ L \left[ e^{xt} \right] = \int_0^\infty e^{-s x} x f(x) e^{-s x} dx = -\frac{d}{ds} \int_0^\infty f(x) e^{-s x} dx \]
\[ = -\frac{d}{ds} L \left[ f \right] \]

4. \[ L \left[ f(x-a) \right] \text{ where } f(x)=0 \text{ for } o < x < a \]

\[ L \left[ f(x-a) \right] = \int_0^\infty f(x-a) e^{-s x} dx \]
\[ = \int_a^\infty f(x-a) e^{-s x} dx \]
\[ = \int_0^\infty f(x) e^{-s (x+a)} dx = e^{-sa} L \left[ f(x) \right] \]

5. Convolution

Define \( (f * g)(x) = \int_0^x f(x) g(x-y) dy \)

Then \[ L \left[ fg \right] = L \left[ f \right] L \left[ g \right] \]

Proof: Just like for Fourier transforms.

III. Examples

A. Solve \( y''(t) + 9y(t) = \cos(3t) \)

\[ y(0)=2, \quad y'(0)=0. \]
Let's try to Laplace transform this equation.

Let \( F(s) = \mathcal{L}[y] \)

\[
\mathcal{L}[y''] = s^2 F(s) - sy(0) - y'(0) = s^2 F(s) - 2S
\]

\[
\mathcal{L} [\cos(3x)] = \int_0^\infty e^{-sx} \cos(3x) \, dx = \frac{1}{2} \int_0^\infty e^{-sx} (e^{3ix} + e^{-3ix}) \, dx
\]

\[
= \frac{1}{2} \left[ \frac{e^{-sx} e^{3ix}}{3i - s} + \frac{e^{-sx} e^{-3ix}}{-(3i + s)} \right]_0^\infty
\]

\[
= \frac{1}{2} \left[ \frac{1}{5 - 3i} + \frac{1}{5 + 3i} \right] = \frac{s}{(5 + 3i)(5 - 3i)}
\]

So, the Laplace transformed equation is

\[
s^2 F(s) - 2S + 9 F(s) = \frac{s}{(5 + 3i)(5 - 3i)}
\]

\[
(s^2 + 9) F(s) = 2S + \frac{s}{(5 + 3i)(5 - 3i)}
\]

\[
F(s) = \frac{2S}{(5 + 3i)(5 - 3i)} + \frac{s}{(5 + 3i)(5 - 3i)^2}
\]

To solve for \( y(x) \), we invert the transform:

\[
\mathcal{L}^{-1} \left[ \frac{1}{2\pi i} \int_{C} F(s) e^{sx} \, ds \right]
\]

How large is \( C \) large enough?

\( F(s) \) must be analytic to the right.

\( > \) all singularities to the left.

poles at \( z_0 = \pm 3i \), \( \text{Re} \ z_0 = 0 \).

So, any \( C > 0 \) will do.

For \( \text{Re} \ C \) close to the contour on the right \( \Delta \text{Re} z = 0 \).
For \( \alpha > 0 \), close the contour on the left & get

\[
\gamma(x) = \frac{2\pi i}{2\pi i} \left( \text{Res}(z;i) + \text{Res}(-3i) \right)
\]

Two terms for each residue, single & double pole

\[
\text{Res}(z;i) = \lim_{s \to 3i} \left( \frac{d}{ds} \left( \frac{5e^{3s}}{(s+3i)^2} \right) + \frac{25e^s}{s+3i} \right)_{s=3i}
\]

Single pole

\[
= \int \frac{e^{3s} + 5e^{s}e^{3s}}{(s+3i)^2} - \frac{25e^{s}}{(s+3i)^3} \bigg|_{s=3i} + \frac{e^{3i}}{s+3i}
\]

Double pole

\[
= e^{3i} \pi \left[ 1 + \frac{1+i}{-36} - \frac{e^{3i}}{6i(-36)} \right]
\]

\[
= e^{3i} \pi \left[ \frac{36-1+i(-3i)}{9} \right] = e^{3i} \pi \left[ 1 - \frac{i \pi}{12} \right]
\]

The residue at \( s = -3i \) is the complex conjugate

\[
\Rightarrow \gamma(x) = e^{3i} + e^{-3i} \pi - \frac{i \pi}{12} (e^{3i} - e^{-3i})
\]

\[
= 2 \cos(3x) + \frac{1}{6} \pi \frac{e^{3i} - e^{-3i}}{2i}
\]

\[
= 2 \cos(3x) + \frac{\pi \sin(3x)}{6}
\]

B. We can also use Laplace transforms to solve integral equations.

Suppose we know that

\[
f(x) = 1 + \int_0^x x e^{-x} f(x-x') \, dx'
\]

where \( f(x) = 0 \) for \( x < 0 \)
more that this implies that \( f(x-x') = 0 \) for \( x' > x \)
so that the above equation can be written

\[
f(x) = 1 + \int_0^x x' e^{x-x'} f(x-x') \, dx'
\]

This is the convolution that acts nicely
with respect to Laplace transforms.

Let \( L[f] = F(s) \).

\[
F(s) = L[1] + F(s) L[x e^{-x}]
\]

\[
L[1] = \int_0^\infty 1 \cdot e^{-sx} \, dx = \frac{1}{s}
\]

\[
L[xe^{-x}] = \int_0^\infty xe^{-x} e^{-sx} \, dx = \frac{-d}{ds} \int_0^\infty e^{-x} e^{-sx} \, dx
\]

\[
= -\frac{d}{ds} \frac{1}{s+1}
\]

\[
= \frac{1}{(s+1)^2}
\]

So,

\[
F(s) = \frac{1}{s} + F(s) \frac{1}{(s+1)^2}
\]

\[
(s+1)^2 F(s) = \frac{(s+1)^2}{s} + F(s)
\]

\[
\frac{(s+1)^2}{s} = F(s) \left[ (s+1)^2 - 1 \right] = F(s) \left[ s^2 + 2s \right] = F(s) \cdot s \cdot (s+2)
\]

\[
F(s) = \frac{(s+1)^2}{s(s^2+2s)} \quad \text{poles at } s = -1, s = -2
\]

Now we do the inverse transform:

\[
f(x) = \frac{1}{2\pi i} \int_{C_c} ds \ F(s) e^{sx} \quad \text{for } c > 0
\]
For \( x > 0 \), closing the contour on the left yields

\[ f(x) = \text{sum of residues} \]

\[ = \Re_2(0) + \Re_2(-1) \]

\[ \Re_2(-1) = \frac{(s-1)^2}{s^2} e^{sx} \bigg|_{s=-2} = \frac{1}{4} e^{-2x} \]

At \( s=0 \)

\[ F(s) e^{sx} = \frac{1}{s^2} \left[ \frac{1 + (s+1)^2}{2(s+1)} \right] \]

\[ = \frac{1}{2s^2} \left[ 1 + (s+x)s + \ldots \right] \left[ 1 - s/2 + \ldots \right] \]

\[ = \frac{1}{2s^2} \left[ 1 + \left( \frac{9}{4} + x \right) s + \ldots \right] \]

\[ = \frac{1}{2s^2} + \left( \frac{9}{4} + x \right) \frac{1}{s} + \ldots \]

\[ \Rightarrow \Re_2(0) = \frac{3}{4} + \frac{x}{2} \]

So

\[ f(x) = \frac{3}{4} + \frac{x}{2} + \frac{1}{4} e^{-2x} = \frac{1}{4} \left[ e^{-2x} + 2x + 3 \right] \]