Lecture # 23: Green's Function Methods

I. Laplace Transforms & The Convolution Method

Last time we talked about using Laplace transforms to solve ODE's. (This works well when the ODE has constant coefficients.)

It turns out that we can avoid doing some of the Laplace & inverse Laplace transforms by using convolutions.

Example: Consider \( y'' + 3y' + 2y = f(t) \) \( y(0) = y'(0) = 0 \).

Let \( L[y] = Y(s) \)

\[
\begin{align*}
L[y] &= sY(s) \\
L[y'] &= sY(s) \\
L[y''] &= s^2Y(s)
\end{align*}
\]

So \( (s^2 + 3s + 2)Y(s) = L(f(t)) \) (Don't evaluate this. We won't need it!)

\[
Y(s) = \frac{L(f(t))}{s^2 + 3s + 2}
\]

Note that

\[
G(t) = L^{-1}\left[ \frac{1}{s^2 + 3s + 2} \right] = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}
\]

is easy to compute.
So, \( Y(s) = L[e^{-x}] L[e^{-x} - e^{-2x}] \)

Recall from last time that such product Laplace transforms arise from convolutions.

\[ \text{I.e., } Y(t) = \int_0^t \delta(t - \tau) f(\tau) \, d\tau \]

Let \( t = 2 - \tau \) and \( \tau = 2 - t \)

\[ = \int_0^2 \delta(t - \tau) f(\tau) \, d\tau \]

\[ = \int_0^2 \delta(2 - t) f(t) \, dt \]

Note that \( \delta \) acts as a Green's function, as discussed last week.

\[ \text{I.e., } Y(t) = \int_0^t \delta(t - \tau) f(\tau) \, d\tau \]

This is not hard to calculate for simple \( f(\tau) \).

II. Laplace transforms & Green's functions

Last week we said that a Green's function gives the response to an impulsive driving force.

How does this work here?

Write \( f(t) = \int_0^\infty \delta(t - \tau) f(\tau) \, d\tau \)

We only consider the system after \( t = 0 \).

Now solve \( G' + 3G + 2G = \delta(t - \tau) \)

\[ L[\delta(t - \tau)] = \int_0^\infty e^{-s\tau} \delta(t - \tau) = e^{-s(t - \tau)} \]
\[ L [g] = \frac{e^{-sz}}{(s+1)(s+2)} \]

\[ g(t-x) = \int_{0}^{t} e^{s \tau} e^{-sz} \, d\tau = e^{-(t-x)} - e^{-2(t-x)} \]

Since the equation is linear, it is clear that

\[ y(t) = \int_{0}^{t} G(t-x) f(x) \, dx \]

forces applied before time \( t \) survive to \( y(t) \). Drivning force first applied here.

III. Other Green’s Functions

Green’s Functions are useful whenever

\( i \) The equation is of the form

\[ Ly = f(t) \]

\( L \) Linear operator

\( f(t) \) Drivning force

\( ii \) you can solve the homogeneous equation \( Ly = 0 \).

I.e., it is useful in cases where Laplace & Fourier transforms may not be. In such cases we can solve \( Lg = g(x-x_0) \) directly.

Example

Consider \( xy'' + 2y' = f(x) \)

The homogeneous equation: \( xy'' + 2y' = 0 \)

is solved by \( y = 1, y = t/x \)

Let’s solve \( xy'' + y' = \delta(x-x_0) \)
For \( x < x_0 \) we must have

\[
G = a \cdot x + b \cdot \frac{1}{x} = c
\]

For \( x > x_0 \) we must have

\[
G = a_1 \cdot x + b_1 \cdot \frac{1}{x} = c_2
\]

for some \( a, b \).

So,

\[
G = c_2 + \delta(x-x_0) [c_2 - c_2]
\]

\[
G(x) = c_2 + \delta(x-x_0) [c_2 - c_2]
\]

\[
= c_2(x) + \delta(x-x_0) [c_2(x) - c_2(x)]
\]

\[
= c_2(x) + \delta(x-x_0) [c_2(x) - c_2(x)]
\]

\[
= c_2(x) + \delta(x-x_0) [c_2(x) - c_2(x)]
\]

\[
+ \delta'(x-x_0) [c_2(x) - c_2(x)]
\]

But is \( \delta(x-x_0) \) on RHS, so need coefficient of \( \delta'(x-x_0) \) to vanish.

\[
\text{Note: we want just } \delta(x-x_0) \text{ on RHS, so need Coefficient of } \delta'(x-x_0) \text{ to vanish.}
\]

But \( \delta' \) only in \( G'' \) \(
\implies x_0 \int [c_2(x) - c_2(x)] = 0
\)

\[
\implies c_2(x) = c_2(x)
\]

\[
G \text{ is continuous at } x_0!
\]
\[ \Rightarrow \quad a_\gamma + b_\zeta x_0 = a_\zeta + b_\gamma x_0 \]

We now get various terms in our equation:

\[ x \zeta'' + \zeta' \quad (= 0 \text{ since } \zeta \text{ solves homogeneous eqn}) \]

\[ + \delta(x-x_0) \left[ x \left( \zeta''(x) - \zeta''(x_0) \right) + \left( \zeta''(x) - \zeta''(x_0) \right) \right] \quad (\approx 0 \text{ since both } \zeta, \zeta'' \text{ solve homogeneous eqn}) \]

\[ + \delta(x-x_0) \left[ x_0 \left( \zeta'(x_0) - \zeta'(x_0) \right) \right] = \delta(x-x_0) \]

Term comes only from \( \zeta' \).

The \( \delta \)-function part of \( \zeta' \) vanishes since \( \zeta \) is continuous at \( x = x_0 \).

So,

\[ 1 = x_0 \left( \zeta'(x_0) - \zeta'(x_0) \right) \]

\[ = x_0 \left( -\frac{b_\zeta}{x_0} + \frac{b_\gamma}{x_0} \right) = -\frac{1}{x_0} \left( b_\zeta - b_\gamma \right) \]

Combining this with continuity we find

\[ \left( a_\gamma - a_\zeta \right) + \left( b_\gamma - b_\zeta \right) \frac{1}{x_0} = 0 \]

or

\[ a_\gamma - a_\zeta = +1 \]

So, we have solved for the change in the coefficients \( a_\gamma, a_\zeta \). But how can we find the precise values?

We need more equations... we need more input!

The point is that there are allowed homogeneous solutions even without any driving force. Thus, we can add any such solution to our \( G(x, x_0) = G_{\zeta}(x) \).
One fixes this by imposing some sort of boundary condition.

**Initial Condition**

This could be an initial condition or we used for Laplace transforms: E.g. Fix \( y(i), y'(i) \)

In this case, first solve the homogeneous equation \( y_0 \):

\[
y_0 = -y'(i) \frac{1}{X} + [y(i) + y'(i)]
\]

Then, write \( y = y_0 + \Delta \). We must have \( \Delta(i) = 0 = \Delta'(i) \).

Impose these (homogeneous) conditions on \( \Delta \):

For \( x_0 < 1 \) this means \( a_0 = 0 \) & \( b_0 = 0 \).

For \( x_0 > 1 \) this means \( a_0 = a \), \( b_0 = b \).

Thus, \( G(x, x_0) = \begin{cases} 
\left( \frac{x_0}{X} - 1 \right) \Theta(x-x_0) & \text{for } x_0 < 1 \\
\left( 1 - \frac{X}{x_0} \right) \Theta(x-x_0) & \text{for } x_0 > 1
\end{cases} \)

Finally, \( y(x) = y_0(x) + \int_{-\infty}^{\infty} dx_0 \Theta(x-x_0) G(x, x_0) \)

**Regularity Condition**

Often the only boundary condition is that the solution does not diverge. E.g., require \( y \) finite at \( x = 0 \) & vanishes as \( x \to \pm \infty \).

Let's assume that \( f(x) \) vanishes for \( x < 0 \). Then we only need \( G(x, x_0) \) for \( x > 0 \).

Then \( G(0, x_0) = G_x(0) = a_0 + b_0 \frac{1}{x} \)

finite \( \Rightarrow b_0 = 0 \).
Note: As you will see in E & M, Greens' Functions are also very useful in solving boundary dimension problems.

As usual, \( \varphi(x) = \int a(x) \cdot g(x,x') f(x') \) since \( y(x) = 0 \) satisfies both advanced & retarded. BCs.

Since \( \Theta(x-x') = (x-x') \Theta(x-x') \),

\[ g(x,x') = (1-x') \Theta(x-x') \]

\[ g(x,x') = 0 \quad \text{for} \ x < x' \]

(e.g. 6) has an initial value \( \Phi(x) \) = 0.

We actually solved these cases above.

Advanced: \( C(x) = 0 \) for \( x > x_0 \).

Retarded: \( C(x) = 0 \) for \( x < x_0 \).

The system does not respond until forced.

Thus \( y(x) = 0 \) & \( b = -1 \).

Thus \( a = 1 \).

\[ g(x) : \begin{cases} 1 & \text{for} \ x > x_0 \\ -1 & \text{for} \ x < x_0 \end{cases} \]

Note. \( y(x) \) is clearly "irregular."