Lecture #24  The Gamma Function

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III. \( \Gamma (z) \) on \( \mathbb{C} \)

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I. Another special function: \( \Gamma (z) \)

Our last topic for this class is Analytic Expressions.

These are somewhat like Taylor Expressions but they do not converge! Nevertheless, they are very useful.

An excellent illustration of such techniques is provided by the so-called Gamma Function. Let us begin with the observation that

\[
\int_0^{\infty} x^ne^{-x} \, dx
\]

Proof: Integrate by parts

\[
\int_0^{\infty} x^n e^{-x} \, dx = (-1)^n \int_0^{\infty} e^{-x} \, dx = (-1)^n \Gamma (n+1)
\]

and for \( n = 0 \), we have \( \int_0^{\infty} e^{-x} \, dx = 1 \)

Note that this expression is also interesting for \( n \notin \mathbb{Z} \).

Define

\[
\Gamma (z) = \int_0^{\infty} x^{z-1} e^{-x} \, dx
\]

So, \( \Gamma (n+1) = n! \) (name the extra 1 on Euler)

\[
\Gamma (z) = (z-1) \Gamma (z-1) \quad \text{for any} \quad z.
\]

For which \( z \in \mathbb{C} \) is \( \Gamma (z) \) defined?

Well, the integral converges (uniformly) for \( \Re (z) > 0 \).

It defines \( \Gamma (z) \) as an analytic function of \( z \) for \( \Re (z) > 1 \).

Since

\[
\Gamma (z+1) = \int_0^{\infty} (z+1-1) x^{z-1} e^{-x} \, dx
\]

Note: \( x^y = e^{y \ln x} \)

\[
|\Gamma (z)| = |\int_0^{\infty} e^{-x} \, dx|\]

This integral always converges at \( \infty \) (due to \( e^{-x} \)).

It may diverge at \( x = 0 \) if \( x + \Re (z) \leq 1 \).

However, we may attempt to define \( \Gamma (z) \) for \( \Re (z) < 1 \)

by analytic continuation.

II. More on Analytic Continuation

Recall that analytic continuation is the idea that we might be able to define some analytic function that agrees with \( \Gamma (z) \) in the region where it is already defined, but which also makes sense elsewhere.

I claimed before that such an analytic extension is unique up to at most a choice of branches.

Proof

Suppose \( f(z) \) & \( g(z) \) are analytic functions on the same connected domain \( D \subseteq \mathbb{C} \) such that
\( f(z) = g(z) \) on a smaller (open) domain \( D \) but \( f(z) \neq g(z) \) at some \( z_0 \in D \)

Then: Connect \( z \) to \( R \) along some curve \( C \) and find the last point \( z_2 \) on \( C \) where \( f(z) = g(z) \).

Note: Since \( f \) & \( g \) are analytic on \( D \), they are both continuous on \( D \) as well. Since they agree to the right of \( z_2 \), they must agree at \( z_2 \) as well.

Similarly, \( \left( \frac{d}{dz} \right)^n f \) & \( \left( \frac{d}{dz} \right)^n g \) are analytic and agree to the right of \( z_2 \), since we may calculate these derivatives along the curve \( C \).

Thus, \( \left( \frac{d}{dz} \right)^n f \bigg|_{z=z_2} = \left( \frac{d}{dz} \right)^n g \bigg|_{z=z_2} \), so that both \( f \) & \( g \) have the same Taylor series expansion about \( z = z_2 \).

Since both \( f \) & \( g \) are analytic, these Taylor expansions converge in some disk of non-zero radius containing \( z_2 \), & converge to \( f \) & \( g \).

Thus, \( f \) & \( g \) also agree to the left of \( z_2 \) along \( C \).

\[ \Rightarrow \quad f = g \text{ everywhere on } D. \]

Example: Consider \( \frac{\Gamma(z)}{\Gamma(z-1)} \).

So far, we have defined this only for \( \Re(z) > 1 \). But there we have

\[ \frac{\Gamma(z)}{\Gamma(z-1)} = z-1 \]

But the RHS is analytic everywhere, so it gives

The analytic extension of \( \frac{\Gamma(z)}{\Gamma(z-1)} \) to \( \mathbb{C} \).

\[ \Rightarrow \quad \Gamma(z) \text{ on } C \]

If we analytically continue \( \Gamma(z) \) then we also continue \( \Gamma(z-1) \) and \( \Gamma(z)/\Gamma(z-1) = z-1 \).

So this relation must hold wherever \( \Gamma(z) \), \( \Gamma(z-1) \) are defined. \( \Leftrightarrow \) \( \Gamma(z) = \frac{1}{2} \Gamma(z+1) (z) \)

We have defined \( \Gamma(z+1) \) for \( \Re(z) > -1 \), and \( (z) \) also defines \( \Gamma(z) \) for \( \Re(z) > -1 \) but there is a simple pole at \( z=0 \).

Recall that \( (z) \) is defined by integrals by parts:

\[ \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx = \int_0^\infty e^{-x} \frac{x^{z-1}}{z} dx \]

We can do the same for \( \Gamma(z+1) = \frac{-1}{z} \Gamma(z+1) + \frac{1}{z} \Gamma(z) \) (why)

We see that \( \Gamma(z+1) \) exists for \( \Re(z) > 0 \) so (why) shows that \( \Gamma(z) \) exists for \( \Re(z) > 0 \) even now singularities yet in this case.
We can repeat this procedure to define $\Gamma(z)$ for all $z \in \mathbb{C}$ except $z = 0, -1, -2, \ldots$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n)} = \frac{\Gamma(z+n)}{\Gamma(n)}$$

$\equiv \Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$

The residue of $\Gamma$ at $z = -n$ is

$$\text{Res} \left(\Gamma, -n\right) = \frac{\Gamma(-n)}{(-n)(-n+1)(-n+2)\cdots(-n+n)} = \frac{(-1)^n n!}{\Gamma(n)}$$

$$\equiv (-1)^n \frac{1}{n!} = \frac{(-1)^n}{n!}$$

For real $x$, $\Gamma(x)$ looks like

\[
\begin{array}{c}
\text{IV. The Beta Function} \\
\text{A related object is the Beta function.}
\end{array}
\]

\[B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt\]

To see why $x \mapsto x^e$,

$$\Gamma(x) = \left[ \int_0^\infty e^{-x^2} \, dx \right] \left[ \int_0^\infty e^{-y^2} \, dy \right]$$

Now, let

$$x = (y^2)^{1/2}, \quad y = (x^{1/2})$$

$$dx = 2x^{1/2} \, dy$$

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

\[\text{Proof:} \]

$$\Gamma(x) \Gamma(1-x) = \Gamma(x+1-x) \Gamma(2, 1-x)$$

$$= B(2, 1-x) = \int_0^1 \frac{2}{x} \left( 1 - x \right)^{1-x} \, dx$$

Consider $-e < \theta < 0$. The integral converges.

Let $u = e^x, \quad du = e^x \, dx = \frac{du}{e^x} = \frac{dx}{u}$,

$$\Gamma(x) \Gamma(1-x) = \int_0^\infty \frac{u^{2-1}}{(1-u)^{x-1}} (1-u)^{1-x} \, du$$

Now, let $u = e^x, \quad du = e^x \, dx$ to find

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$
This is an integral that you did on your hw:

while back by considering

\[ I' = \int_{0}^{\infty} \frac{e^{2x}}{1 + e^x} \, dx \]

RESULT:

\[ \Pi(\frac{\pi}{2}) \Pi(1-\frac{\pi}{2}) = \int_{0}^{\infty} \frac{e^{2x}}{1 + e^x} \, dx = \frac{\pi}{\sin(\pi \theta)} \]

A frequently used special case is

\[ \Pi\left(\frac{\pi}{4}\right) \Pi\left(\frac{\pi}{4}\right) = \frac{\pi}{\sin(\pi \theta)} = \frac{\pi}{2} \]

i.e., \[ \Pi\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{2}} \]