Lecture #24: The Gamma Function

I. Another special function: \( \Gamma(z) \)

II. More on Analytic Continuation

III. \( \Gamma(z) \) on \( \mathbb{C} \)

IV. The Beta function

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I. Another special function: \( \Gamma(z) \)

Our last topic for this class is **Asymptotic Expansions**. These are somewhat like Taylor Expansions but they do not converge! Nevertheless, they are very useful.

An excellent illustration of such techniques is provided by the so-called **Gamma function**. Let us begin with the observation that

\[
 n! = \int_0^\infty x^n e^{-x} \, dx
\]

**Proof:** Integrate by parts

\[
 \int_0^\infty x^n e^{-x} \, dx = (-1)^n \left[ \int_0^\infty e^{-x} \, dx \right]^n
\]

\[
 = n \int_0^\infty x^{n-1} e^{-x} \, dx
\]

and for \( n = \infty \), we have \( \int_0^\infty e^{-x} \, dx = 1 \)

Note that this expression is also interesting for \( n \notin \mathbb{Z} \).

Define

\[
 \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx
\]
So, \( \Gamma(n+1) = n! \) (Blame the extra 1 on Euler)

\[ \Gamma(z) = (z-1) \Gamma(z-1) \quad \text{for any } z. \]

For which \( z \in \mathbb{C} \) is \( \Gamma(z) \) defined?

Well, the integral converges (uniformly) for \( \text{Re}(z) > 0 \) & defines \( \Gamma(z) \) as an analytic function of \( z \) for \( \text{Re}(z) > 1 \).

Since

\[ \Gamma'(z) = \int_0^\infty \left( \frac{z-1}{\mathbb{e}^x - 1} \right) \mathbb{e}^{-x} \]

Note: \( \mathbb{e}^{z+i\gamma} = \mathbb{e}^z \mathbb{e}^{i\gamma} \mathbb{e}^{-x} \)

\[ \Rightarrow |z|^z = \mathbb{e}^x. \]

This integral always converges at \( x = 0 \) (due to \( e^{-x} \))

may diverge at \( x = 0 \) if \( x = \text{Re}(z) \leq 1 \).

However, we may attempt to define \( \Gamma(z) \) for \( \text{Re} z \leq 1 \)

by analytic continuation.

II. More on Analytic Continuation

Recall that analytic continuation is the idea that we might be able to define some analytic function that agrees with \( \Gamma(z) \) in the region where it is already defined, but which also makes sense elsewhere.

I claimed before that such an analytic continuation is unique, up to at most a choice of branch cuts.

\[ \textbf{Proof} \]

Suppose \( f(z) \) & \( g(z) \) are analytic functions on the same connected domain \( D \subseteq \mathbb{C} \) such that ...
\[ f(z_0) = g(z_0) \text{ on a smaller (open) domain } \mathbb{D} \]

but \[ f(z_0) \neq g(z_0) \text{ at some } z_0 \in \mathbb{D} \]

Then: Connect \( z_0 \) to \( \mathbb{R} \) along some curve \( C \)

and find the last point \( z_1 \) on \( C \)

where \( f(z) = g(z) \).

Note: Since \( f \) & \( g \) are analytic on \( \mathbb{D} \),

they are both continuous on \( \mathbb{D} \) as well.

Since they agree to the right of \( z_1 \), they

must agree at \( z_1 \) as well.

Similarly, \( \left( \frac{d}{dz} \right)^n f \) & \( \left( \frac{d}{dz} \right)^n g \) are analytic

and agree to the right of \( z_1 \); since we may

calculate these derivatives along the curve \( C \).

Thus, \( \left( \frac{d}{dz} \right)^n f \bigg|_{z = z_1} = \left( \frac{d}{dz} \right)^n g \bigg|_{z = z_1} \),

so that both \( f \) & \( g \) have the same Taylor's series expansion about \( z = z_1 \).

Since both \( f \) & \( g \) are analytic, these Taylor expansions converge in some disk of non-zero radius containing \( z_1 \), & converge to \( f \) & \( g \),

Thus, \( f \) & \( g \) also agree to the left of \( z_1 \) along \( C \).

This is a contradiction, so the assumed point \( z_0 \) where \( f \) \& \( g \) disagree

cannot exist.

Instead, \( f = g \) everywhere on \( \mathbb{D} \).
Example Consider \[ \frac{\Gamma(z)}{\Gamma(z-1)} . \]

So far we have defined this only for \( \text{Re}(z) > 1 \). But there we have
\[
\frac{\Gamma(z)}{\Gamma(z-1)} = z-1
\]

But the RHS is analytic everywhere, so it gives the analytic extension of \( \frac{\Gamma(z)}{\Gamma(z-1)} \). \( \Box \)

III. \( \Gamma(z) \) on \( \Delta \)

If we analytically continue \( \Gamma(z) \) then we also continue \( \Gamma(z-1) \) and \( \Gamma(z)/\Gamma(z-1) = z-1 \).

So this relation must hold wherever \( \Gamma(z), \Gamma(z-1) \) are defined. \( \Rightarrow \) \( \Gamma(z) = \frac{1}{z-1} \Gamma(z+1) \) (**)

We have defined \( \Gamma(z) \) for \( \text{Re}(z) > 1 \).

So (**), also defines \( \Gamma(z) \) for \( \text{Re}(z) > 1 \).

Note that there is a simple pole at \( z=0 \). Recall that (**), is obtained by integrating by parts:
\[
\Gamma(z) = \int_0^\infty x^{z-1}e^{-x} \, dx = \int_0^\infty \frac{x^z}{e^x} \, dx
\]

We can do the same for \( \Gamma'(z) = -\frac{1}{z} \Gamma(z+1) + \frac{1}{2} \Gamma'(z+1) \) (***)

We saw that \( \Gamma'(z+1) \) exists for \( \text{Re}(z) > 0 \) so (***) shows that \( \Gamma'(z) \) exists for \( \text{Re}(z) > 0 \) [no new singularities yet in this case.]
We can repeat this procedure to define \( \Gamma (z) \) for all \( z \in \mathbb{C} \) except \( z = 0, -1, -2, \ldots \)

\[
\Gamma (z) = \frac{\Gamma (z+n)}{z(z+1)(z+2)\cdots(z+n-1)} = \frac{\Gamma (z+n+1)}{z(z+1)\cdots(z+n)}
\]

\( \Rightarrow \) \( \Gamma (z) \) has simple poles at \( z = 0, -1, -2, \ldots \)

The residue of \( \Gamma \) at \( z = -n \) is

\[
\text{Res}_{z=-n} \Gamma (z) = \frac{\Gamma (z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)} \bigg|_{z=-n} = (-1)^n \frac{\Gamma (1)}{n!} = (-1)^n \frac{1}{n!}
\]

For real \( x \), \( \Gamma (x) \) looks like

\[
\begin{aligned}
\text{IV. The Beta function} \\
\text{A related object is the Beta function}
\end{aligned}
\]

Both \( \Gamma (z) \) \& \( B(a,b) \) come up frequently in probability & statistics, \& in the normalization of wave functions in quantum mechanics.

\[
B(a,b) = \frac{\Gamma (a) \Gamma (b)}{\Gamma (a+b)} = \int_0^1 x^{a-1} (1-x)^{b-1} \, dx
\]

To see this, write

\[
\tau (a) \tau (b) = \left[ \int_0^\infty x^{a-1} e^{-x} \, dx \right] \left[ \int_0^\infty y^{b-1} e^{-y} \, dy \right]
\]
Now let $x = (y'), y = (y')'$

\[ dx = 2y'y' \, dy \quad dy = 2y'y' \, dx \]

\[ \Gamma(a) \, \Gamma(b) = 4 \int_0^\infty e^{-(x'y')^2} (x'y')^{2a-1} \, dx' \int_0^\infty e^{-(y'y')^2} (y'y')^{2b-1} \, dy' \]

Change from $x'y'$ to polar coordinates $r, \theta$ \hspace{1cm} $dx'dy' = r \, dr \, d\theta$

\[ \Gamma(a) \, \Gamma(b) = 4 \int_0^{\pi/2} \int_0^\infty \, r \, e^{-r^2} (r \cos \theta)^{2a-1} (r \sin \theta)^{2b-1} \, dr \, d\theta \]

\[ = 4 \int_0^{\pi/2} \int_0^\infty \, r^{2(a+b)-1} \, e^{-r^2} \, dr \, \cos^{2a-1} \, \sin^{2b-1} \, d\theta \]

Let $u = r^2$, $\mu = \sin^2 \theta$, $1-\mu = \cos^2 \theta$

\[ 2r \, dr = du, \quad d\theta = 2 \sin \theta \, \cos \theta \]

\[ \Gamma(a) \, \Gamma(b) = \left[ \int_0^\infty \, u^{a+b-1} \, e^{-u} \, du \right] \left[ \int_0^1 \, \mu^{a-1} \, (1-\mu)^{b-1} \, d\mu \right] \]

\[ = \Gamma(a+b) \, \beta(a, b) \]

With a bit of work, one can use this to show that

\[ \Gamma(a) \, \Gamma(1-a) = \frac{\pi}{\sin (\pi a)} \]

**Proof**

\[ \Gamma(a) \, \Gamma(1-a) = \Gamma(a) \, \Gamma(a+1-a) \, \beta(2, 1-a) \]

\[ = \beta(2, 1-a) = \int_0^1 \, x^{a-1} \, (1-x)^{-2} \, dx \]

Consider $0 < a < 1$ where this integral converges

\[ \Gamma(a) \, \Gamma(1-a) = \int_0^\infty \, \frac{du}{(1+u)^a} \, \frac{u^{b-1}}{(1+u)^{b-1}} \, du = \int_0^\infty \frac{u^{b-1}}{(1+u)} \, du \]

Let $u = e^x$, $du = e^x \, dx$ \hspace{1cm} $x \rightarrow -\infty$ and

\[ \Gamma(a) \, \Gamma(1-a) = \int_0^\infty \frac{e^{2x}}{1+e^x} \, dx \]
This is an integral that you did on your HW a while back by considering

\[ I = \int \frac{e^{2x}}{1 + e^{2x}} \, dx \]

Result:

\[ \prod_{1}^{n} \gamma (1/2 - z) = \int_{0}^{\infty} \frac{e^{2x}}{1 + e^{2x}} \, dx = \frac{\pi}{\sin(\pi z)} \]

A frequently used special case is

\[ \gamma (1/2) \gamma (1/2) = \frac{\pi}{\sin(\pi/2)} = \pi \]

i.e. \[ \gamma (1/2) = \sqrt{\pi} \]