Lecture #26  More Asymptotic Series

I. Asymptotic series Review

II. Asymptotic series via integration by parts

III. steepest descent or stationary phase?

I. Asymptotic series Review

Last time we discussed the idea that some series expansions can be useful even when they don't converge.

Here's a definition:

A formal power series \( \sum_{n=0}^{\infty} c_n z^{-n} \) is asymptotic to \( h(z) \) if

\[
\lim_{z \to \infty} \frac{|h(z) - \sum_{n=0}^{\infty} c_n z^{-n}|}{\theta(z)} = 0
\]

We write \( h(z) \sim \sum_{n=0}^{\infty} c_n z^{-n} \) [not "equal"]

This might hold under certain conditions (e.g., \( \Re(z) > 0 \))

This readily generalizes to fractional powers.

Note: The asymptotic series for \( h(z) \) [in a given region] is unique.

However, we can have \( h_1(z) \sim \sum_{n=0}^{\infty} c_n z^{-n} \sim h_2(z) \) for \( h_1(z) \neq h_2(z) \)

E.g. For \( \Re(z) > 0 \)

\[
e^{-z} \sim \sum_{n=0}^{\infty} c_n z^{-n} \quad \text{with} \quad c_n = 0
\]

So \( e^{-z} \sim 0 \)
II. Asymptotic Series via Integration by Parts

Last time we talked about generating an asymptotic series using the method of steepest descents. I'd like to return to that method in a bit, but let me first address another method that is sometimes useful and that appears on your homework.

Sometimes you can generate an asymptotic series by doing integrations by parts.

A classic example of this is the "error function."

Recall that Gaussians often come up in probability theory and are at least a decent model of statistical errors in many experiments.

As a result, one is often interested in the "error function"

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]

or, the "complimentary error function"

\[ \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \]

How shall we integrate this by parts?

Could do:

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \left( x e^{-x^2} \right)_x^\infty - \frac{2}{\sqrt{\pi}} \int_x^\infty (dxe^{-x^2}) dx \]
\[
= -\frac{2}{\sqrt{\pi}} x e^{-x^2} + \frac{4}{\sqrt{\pi}} \int_{x}^{\infty} x^2 e^{-x^2} \, dx
\]

But note: For large \(x\), this is larger than \(x\) itself. Not negligible!

So this was not useful... Instead, let's do something else.

\[
d_{x} e^{-x^2} = 2x e^{-x^2}
\]

So,

\[
e^{-x^2} = -\frac{1}{2x} d_{x} e^{-x^2}
\]

I.e.,

\[
\text{erfc} = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \left( -\frac{1}{2x} \right) e^{-t^2} \, dt
\]

\[
= -\frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \bigg|_{x}^{\infty} + \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} \left( 2x t^{-2} \right) e^{-t^2} \, dt
\]

\[
= -\frac{e^{-x^2}}{\sqrt{\pi} x} + \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} \frac{e^{-t^2}}{t^2} \, dt
\]

Note

\[
\frac{1}{\sqrt{\pi}} \int_{x}^{\infty} \frac{e^{-t^2}}{t^2} \, dt \leq \frac{1}{\sqrt{\pi} x} \int_{x}^{\infty} \frac{e^{-t^2}}{t^2} \, dt
\]

\[
\leq \frac{C \text{erf}(x)}{2 - x^2}
\]

So we have

\[
\text{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi} x} \left( 1 + O(x^{-2}) \right)
\]

We can repeat this trick to generate a full asymptotic series for \(\text{erfc}\).

This trick often works for integral expressions \(\int_{x}^{\infty} f(t) \, dt\).
III. Steepest Descent or Stationary Phase?

I'd now like to go back to the steepest descent method that we used on the $\Gamma$-function last time.

It is very common that interesting expressions are of the form

$$I(\alpha) = I(0) \int_{0}^{\infty} e^{-\alpha S(z)} \, dz$$

where $S(z)$ is independent of $\alpha$.

For the case we studied before, $S(z)$ was real on the real axis & $S(z) \to \infty$ as $z \to \pm \infty$ along the real axis.

We noted that the integral was dominated by the region near the minimum of $S(z)$, and that in the $\alpha \to 0$ limit we could approximate the integrand as a Gaussian by expanding $S(z)$ about this minimum.

In fact, a similar trick can work even if $S(z)$ has no minimum on the real line!

To understand this, suppose that $S(z)$ is an analytic function. Then we are free to deform the contour of integration (taking care to avoid singularities). If we can find a contour $\gamma$ such that $S(z)$ has a "minimum" at some $z_0$ along $\gamma$,

Then we can approximate $I(\alpha)$ by expanding $S(z)$ about $z_0$. 
Note: Since \( S(z) \) is analytic at \( z_0 \), \( z_0 \) cannot be a minimum of \( S(z) \) in \( C \). [Recall that analytic functions do not have maxima or minima since their value at any \( z_0 \) is the average of the values over any circle centered at \( z_0 \).]

But there can be a point \( z_0 \) such that \( S(z_0) \) is the "minimum" value that \( S(z) \) takes on \( C \).

Hold on though: if we think of \( S(z) \) as an analytic function on \( C \), it is complex valued. So, what precisely do we mean by a "minimum"? Note that minimizing \( |S(z)| \) is not sufficient, as the sign of \( S(z) \) is crucial!

It turns out that this idea is only useful when the contour \( C \) is chosen to make \( S(z) \) real on \( C \) at least near \( z_0 \).

In this case we really do mean that \( z_0 \) is a local minimum of \( S(z) \) along \( C \).

Note that if \( z \) \( \in \) the derivative of \( S(z) \) along the curve \( C \) must be zero. But if \( S(z) \) is analytic, then \( dS \) is the same in all directions!

So, \( S'(z_0) = 0 \), so \( z_0 \) is a stationary point of \( S(z) \).

Let's look at an example.

[This is a result from one of my recent research projects and has to do with the creation of an infinite universe.]

Suppose that one wishes to Fourier transform \( \frac{1}{\cosh x} \).

Then \( \mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\cosh k} = \int_{-\infty}^{\infty} dx \frac{e^{ikx}}{\cosh x} \).
Let's write this as \[ f(k) = \int_{-\infty}^{\infty} dx \ e^{-S(x)} \]

where \[ S(x) = -ikx + \ln \cosh x. \]

Note that \( k \) is not just an overall factor in front of \( S(x) \) .... but we will see that it is approximately so that this can be ok!

Set \( S'(z_0) = 0 \). \( z_0 \) is a point of "stationary phase"

\[
0 = \frac{d}{dz} \left( -ikz + \ln \cosh z \right) = -ik + \frac{\sinh z}{\cosh z},
\]

\[ z = z_0, \quad k = \tanh(z_0) \]

We want to save this for any \( k \in \mathbb{R} \). To do so, let \( t_0 = -z_i \)

\[
k = \frac{\tanh(i\tilde{z})}{\tilde{z}} = \frac{1}{\tilde{z}} \left( \frac{e^{i\tilde{z}} - e^{-i\tilde{z}}}{e^{i\tilde{z}} + e^{-i\tilde{z}}} \right) = \frac{\sin(\tilde{z})}{\cos(\tilde{z})} = \tan(\tilde{z})
\]

Recall that \( \tan(\tilde{z}) \) takes on all real values for real \( \tilde{z} \) with \( -\pi/2 < \tilde{z} < \pi/2 \). So there is a solution in this interval.

This is good, because \( S(z) \) is singular where \( \cosh(z) = 0 \), \( z = \pm i \pi/2 \).
Since \( z_0 \) is closer to the real axis than is the singularity, it is no problem to deform the contour to pass through \( z_0 \).

But does \( C \) satisfy our requirement to make \( S(z) \) real near \( z_0 \) (with a \( \min \) at \( z_0 \), not a \( \max \))?

To find out, let's expand \( S(z) \) near \( z_0 \),

\[
S(z) = S(z_0) + \frac{1}{2} S''(z_0) (z - z_0)^2 + \frac{1}{6} S'''(z_0) (z - z_0)^3 + \cdots
\]

In our case,

\[
S''(z) = \frac{d}{dz} \left( \frac{\sinh(z)}{\cosh(z)} \right) = \frac{1}{\cosh^2(z)} \left( \cosh^2(z) - \sinh^2(z) \right) = \frac{1}{\cosh^2(z)}
\]

But it's better to write this as

\[
S''(z) = \frac{1}{\cosh^2(z)} = \frac{1}{(e^{-z} + 1)^2}.
\]

\[
e^{-S(z)} = e^{-S(z_0)} e^{-S''(z_0)} (z - z_0)^2/2
\]

So, indeed, \( S''(z) (z - z_0)^2 \) is real on \( C \) and \( S''(z) (z - z_0)^2 > 0 \) on \( C \) if \( C \) is parallel to the real axis.

Thus we have

\[
S(1) \approx e^{-S(z)} \int_{C} e^{-(e^{-z}(z - 2z_0)^2)} \, dz
\]

So

\[
g(1) \approx e^{-S(z_0)} \sqrt{\frac{2\pi}{e^{2z_0} + 1}}
\]
Now,

\[
e^{-5(z)} = \frac{e^{-kz}}{\cosh(2z)} = \frac{e^{-kz}}{\cos(z)} = \frac{e^{-k\tan^{-1}(k)}}{\cos(k\tan^{-1}(k))} = \sqrt{k \tan^{-1}(k)} e^{-k \tan^{-1}(k)}
\]

So,

\[
\mathcal{F}(k) \approx \sqrt{\pi k} e^{-k \tan^{-1}(k)} \quad \text{for} \quad k \ll 1
\]

\[
\approx \sqrt{\pi k} e^{-k} \quad \text{for} \quad k \gg 1
\]

**Note 1:** The choice of \( C \) was entirely determined by \( S''(\omega) \).

We chose \( C \) such that

1. \( S''(\omega) (\omega - \omega_0)^2 / 2 \) is real and
2. \( \frac{S''(\omega)}{(\omega - \omega_0)^2 / 2} > 0 \).

In particular, the value of \( S \) at \( \omega_0 \) did not affect our choice of \( C \), as it just gives an overall factor \( e^{-S(\omega)} \) in front of the Gaussian integral.

**Note 2:** Our final integral was of the form

\[
\int_C dz \ e^{-S''(\omega) (\omega - \omega_0)^2 / 2}
\]

with \( C \) parallel to the real axis because \( S''(\omega_0) \) was real and positive. So, for \( \omega = \omega - \omega_0 \), our integral was

\[
\int_R d\omega \ e^{-S''(\omega) \omega^2 / 2}
\]

Here \( d\omega = dz \). Note, however, that since we want \( \omega \) to take real values if our contour for \( z \) were not parallel to the real axis, then \( d\omega \neq dz \), in that case the integral would pick up an additional phase factor.
Note 3: The approximation is valid when the Gaussian is narrow in comparison with the correction terms.

If \( S \) had been of the form \( k \exp(2x) \) with \( S \) independent of \( k \), this would follow automatically in the limit of large \( k \). But since our \( S \) is not of that form, we must check directly.

The corrections come from the higher terms

\[
\frac{1}{3!} \cdot 5'''(2x) \omega^3 + \frac{1}{4!} \cdot 5''''(2x) \omega^4 + \cdots
\]

The question is whether or not these are important in the region that contributes to the Gaussian integral.

That region is

\[
15''(2x)/\omega^3 \ll 1 \quad \text{for} \quad (k^2+1)\omega \ll 1 \quad \text{in our case.}
\]

We will need

\[
5''(2x) = \frac{\sinh(2x)}{\cosh^3(2x)} = \frac{1}{\cosh(2x)} \cdot \tanh(2x) = \frac{1}{2} \cdot 5''(2x)
\]

Since \( 5''(2x) \omega^3 \) grows with \( \omega \), let us evaluate it at \( \omega = \frac{1}{\sqrt{k^2+1}} \)

we find

\[
5''(2x) (k^2+1)^{-3/2} = \frac{k}{Nk^{3/2}}
\]

we want

\[
\frac{k}{Nk^{3/2}} \ll 1 \quad \Rightarrow \quad \text{our approximation is good asymptotically in the limit}
\]

\( k \to \infty \).

We can, however, get something out of our calculation for large \( k \) as well. For \( k \gg 1 \),

\[
\frac{k}{Nk^{3/2}} \approx \frac{1}{Nk^{3/2}}
\]

So the corrections are of the same order as that given by the stationary phase approximation. The important point is that they are not parametrically larger.
Thus it is reasonable to expect $g(k) \sim C e^{-k^{3/2}}$ for large $k$, where $C$ is an unknown but $k$-independent constant. I.e., $g(k) = O(e^{-k^{3/2}})$. 