Lecture #2: Functions of a Complex Variable

I. Single-valued Functions

II. Multi-valued Functions

III. Branch Cuts

I. Single-valued Functions

We will be interested in complex functions \( f(z) \) of the complex variable \( z \). One often denotes this by \( f: \mathbb{C} \rightarrow \mathbb{C} \).

Any such \( f: \mathbb{C} \rightarrow \mathbb{C} \) may be written in the form

\[
f(z) = u(x, y) + i \cdot v(x, y)
\]

where \( u \) and \( v \) are real functions.

\[
u, v: \mathbb{R}^2 \rightarrow \mathbb{R}
\]

Thus, one may think of \( f \) as a map from the \((x, y)\) plane to the \((u, v)\) plane. We can visualize \( f \) as a map of points, curves, etc.

If for each \( z \) there is one and only one image point \( f(z) \), then \( f(z) \) is single valued.

Example 1: Single-valued Functions

- Complex conjugation: \( f(z) = \overline{z} = x - iy \)
  
  \[
u(x, y) = x
  \]
  
  \[
v(x, y) = -y
  \]

(Reflects across real axis)
**Example 2:** $f(z) = e^z$ (do not confuse this with $z = r e^{i \theta}$)

\[
f(z) = e^z = e^{x+iy} = e^x e^{iy} \quad \text{This expresses } f(z)
\]

\[
\lvert y \rvert \quad \text{in polar form}
\]

\[
f(z) = e^x
\]

\[
\text{arg } (f(z)) = y
\]

\[
\Rightarrow e^z = e^x (\cos y + i \sin y)
\]

more that $f(z) = e^z$ is single-valued. It takes many values of $z$ to the same value $f(z)$ (many-to-one). Example: $f(z') = f(z)$ for $z' = z + i(2\pi n)$ for $n \in \mathbb{Z}$ (i.e., for $n = 0, 1, 2, 3, \ldots$)

But each $z$ defines a unique $f(z)$.

Other examples of many-to-one single-valued functions include powers of $z$: $f(z) = z^2, z^3, z^4, \ldots$

**Example 3:** Related Examples (also single-valued & many-to-one)

Trigonometric & hyperbolic Trig functions

\[
\sin z, \cos z, \sinh z, \cosh z
\]

Definitions

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}
\]

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}
\]

Can be derived from power series expansions

\[
\sinh z = \frac{(e^z - e^{-z})}{2}
\]

\[
\cosh z = \frac{(e^z + e^{-z})}{2}
\]
We see immediately that
\[
\sin(i \cdot z) = -\frac{1}{i} \sinh(z) = i \sinh(z)
\]
and
\[
\cos(i \cdot z) = \cosh(z)
\]

It is straightforward to evaluate these functions at complex arguments, since we know how to evaluate exponentials of complex numbers.

Example: \( \sin(i \cdot z) = \frac{1}{2i} (e^{i \cdot z} - e^{-i \cdot z}) = \frac{1}{2i} (e^z - e^{-z}) \)

\[
= \frac{1}{2i} \frac{-i}{e^z} (e^z - e^{-z}) = \frac{i}{2} (e^{-z} - e^z)
\]

In \( u + iv \) form, \( u = 0 \), \( v = \frac{1}{2} (e^{-i} - e^i) \)

In \( re^{i \theta} \) form, \( r = \frac{1}{2} (e^{-i} - e^i) \), \( \theta = \pi/2 \)

\[
\frac{1}{2} (e^{-i} - e^i)
\]

II. Multi-valued functions

Typical examples are roots & logs.

(i.e., the inverse of many-to-one functions)

Note: \( \log \) will always mean natural \( \log \) \( (\log_2 \) will \( \log \) \( 2 \))

We'll write \( \log_{10} \) for the base 10 logarithm.

In prior form

\[
z = re^{i \theta} = re^{i(\theta + 2\pi n)} \quad \text{for any } n \in \mathbb{Z}
\]

So it vary \( n \) continuously (through \( \mathbb{R} \)) from 0 to \( n \in \mathbb{Z} \)

Find

\[
\log z = \log (r e^{i(\theta + 2\pi n)}) = \log r + \log (e^{i(\theta + 2\pi n)})
\]

\[
= \log r + i(\theta + 2\pi n)
\]

Example 1:

Say consider

\[
\log z = \log (r e^{i(\theta + 2\pi n)}) = \log r + \log (e^{i(\theta + 2\pi n)})
\]

\[
= \log r + i(\theta + 2\pi n)
\]
There are an infinite set of possible values of \( \log z \). It is multi-valued.

(For, one-to-many)

Example 2: Another example is a root, say

\[ f(z) = \sqrt{z} = z^{1/2} \]

To test if a function is multi-valued put in

\[ z = r e^{i(\alpha + 2\pi n)} \]

Which describes one value of \( z \).

And see if you get many roots in the unit plane.

We have

\[ f(z) = \sqrt{z} = \left[ r e^{i(\alpha + 2\pi n)} \right]^{1/2} \]

\[ = r^{1/2} e^{i\alpha/2} e^{i\pi n} \]

\[ = 1 \quad \text{for} \quad n = 0, \pm 2, \pm 4, \ldots \]

\[ = -1 \quad \text{for} \quad n = \pm 1, \pm 3, \ldots \]

\( \Rightarrow \) \( \sqrt{z} \) is double-valued, it has 2 possible values for each \( z \).

In general, \( \sqrt{z} \) can take \( m \) different values for \( m \in \mathbb{Z} \),

\[ \sqrt{z} = r^{1/m} e^{-i\beta/m} e^{i\pi n/m} \quad \text{for} \quad n \in \mathbb{Z} \]

gives \( m \) values on a circle.
III. Branch Cuts (take 1)

clearly, multi-valued functions are problematic:
which value do we take?

we can make a choice. e.g., consider the branch
of the function $\sqrt{z}$ on which $\sqrt{1} = 1$, what is
$\sqrt{z}$ at other values? makes sense to choose $\sqrt{z}$ as
continuous as possible. but as above, problems arise if
we circle the origin. so, we often introduce a
barrier (called a "branch cut") to restrict the domain.

Example: Define $f(z) = \sqrt{z}$ on the domain $\mathbb{C} - \{\text{negative real axis}\}$

Then a unique definition of $\sqrt{z}$ is determined at all such $z$
by analyticity (and in particular by continuity).

In particular,

for $z = r e^{i \theta}$ with $\pi > \theta > -\pi$

$\sqrt{z} = r^{1/2} e^{i \theta/2}$

$e^{-i \pi/4} = f(e^{i \pi/2}) = e^{i \pi/4}$

$e^{-i \pi/4} = f(e^{-i \pi/2}) = e^{-i \pi/4}$
The point: Given a multi-valued function we can always define a single-valued function by introducing branch cuts to restrict the domain & choosing a branch.

There are, however, many such choices. In particular, there is much arbitrariness in choosing the branch cut. For $\sqrt{z}$, any curve from $z=0$ to $z=\infty$ would do.

Example:

Summary: Each choice defines a different single-valued function from the same multi-valued function. The trick is to make the right choice for a given application.

Ok, your turn: What are the answers to the following exercises?

Exercise 1:

\[ f(z) = \sqrt{z} \]

A) \[ f(1) = 1 \]

B) \[ f(-1) = -1 \]

Find: \[ f(-i) = -i \] \[ f(i) = i \]

since \[ f(z) = r^{1/2} e^{i \theta/2} \] for \[ \pi/2 < \theta < \pi \]

Exercise 2:

\[ f(z) = z^{1/2} \]

\[ f(1) = 1 \]

Find: \[ f(z) \], \[ f(-z) \]

Answer: \[ f(z) = r^{1/2} e^{i \theta/2} \] for \[ \pi/4 > \theta > -\pi/4 \]

\[ f(i^2) = f(e^{i \pi/2}) = e^{i \pi/4} \]

\[ f(-i) = f(e^{i 3\pi/2}) = e^{i \pi/2} = i \]
EX 3

A) \( f(2) = \sqrt{2} \) and \( f(1) = 1 \)

- \( f(\pi) = -\sqrt{2} \)
- \( f(3) = \sqrt{3} \)
- \( f(4) = -2 \)

B) \( f(2) = \sqrt[3]{2} \) & \( f(0) = 1 \)

- \( f(\pi) = 2^{\pi/3} e^{i2\pi/3} \)
- \( f(3) = 3^{1/3} e^{i\pi/3} \)
- \( f(4) = 4^{1/3} \)