Lecture #8: Laurent Series Examples

1. \( f(z) = \frac{1}{z-1} \)  
   Expand around \( z = 1 \):  
   \( f = \frac{1}{z-1} \)  
   (Done)

2. \( f(z) = \frac{1}{2z-1} = \frac{1}{2} \left( \frac{1}{z-1/2} \right) \)  
   \( C_{-1} = \frac{1}{2} \)

3. \( f(z) = \frac{1}{z^{1/2}} \)  
   \( \partial \chi \)  
   \( 2\pi \)  

\[ f(z) = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n \]

\[ f(z) = \frac{1}{z-1} = \frac{1}{z} - \frac{1}{z-1} + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n \]

\[ f(z) = \frac{1}{z} - \frac{1}{z} \left( \frac{1}{z-1} \right) + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n \]

\[ \int_{C} f(z) \, dz = 2\pi i \sum_{n=-1}^{\infty} C_n = 2\pi i \left( \frac{1}{2} \right) \]

\[ C_{-1} = \frac{1}{2} \]

\[ C_n = -\left( \frac{1}{2} \right)^{n+1} \]

\[ C_0 = -\frac{\pi}{2} \left( \frac{\pi}{2} \right) = +\frac{\pi}{4} \]

\[ \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n \]

**Note:** We used the Taylor's series for \( \frac{1}{1+z} \) which converges

for \( \left| z - \frac{1}{2} \right| < 2 \)

i.e., "outside the poles"

One can also find a different Laurent series \( \sum_{n=0}^{\infty} (z-1)^n \)

(i.e., expanded about \( z=1 \)) which converges for \( \left| z - 1 \right| > 2 \)

i.e., "outside both poles". The terms in the series change when \( \left| z - 1 \right| = \text{distance from } \frac{1}{2} \) to a singularity
\[
\frac{1}{z-\alpha} = \frac{1}{z+\alpha} = \frac{1}{z-\alpha} \left( \frac{1}{z-\alpha} + \frac{1}{z-\alpha} \right) \\
= \frac{1}{(z-\alpha)^2} \left( \frac{1}{z-\alpha} \right) = \frac{1}{(z-\alpha)^2} \sum_{n=0}^{\infty} (-2\alpha)^n (z-\alpha)^n \\
= \sum_{n=0}^{\infty} (-2\alpha)^n (z-\alpha)^n \\
= \sum_{n=0}^{\infty} \frac{(-2\alpha)^n}{(z-\alpha)^{n+2}} \\
\]

\[e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{1}{z})^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

Note: In no case did I actually find the Laurent series via the integral formulas above. [You can do that, but it is usually hard...]

This case where \( C_n \to 0 \) for all \( n \geq 0 \) is called an "essential singularity".

\[ e^{1/z} \] expanded about \( z = 0 \)?

How to begin...

Require single-valued analytic function, so must introduce branch cut. Ex.

\[ e^{1/z} \]

Oops! Due to branch cut there is no circle centered on \( z = 0 \) on which \( f(z) \) is analytic \( \Rightarrow \) no Laurent series.