Physics 101 Homework 1 Solutions

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1 Ch. 2, §4, p. 51, 13

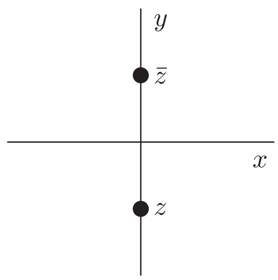
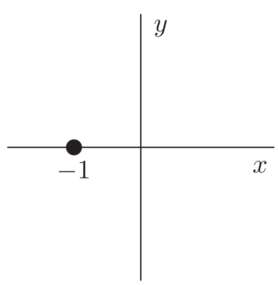


Figure 1: $(x, y) = (0, -1), -i, (r, \theta) = (1, 3\pi/2), \exp(3\pi i/2) = cos(3\pi/2) + isin(3\pi/2)$

2 Ch. 2, §5A, p. 52, 6

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{1+i}{1-i}\right)^2 \left(\frac{1+i}{1+i}\right)^2 = \frac{-4}{4} = -1 \tag{2.1}$$



3 Ch. 2, §5B, p. 53, 25

$$\left(\frac{z_1}{z_2}\right)^* = \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^* = \left(\frac{r_1}{r_2} e^{i(\theta_i - \theta_2)}\right)^* = \frac{r_1}{r_2} e^{-i(\theta_1 - \theta_2)} = \frac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}} = \frac{\bar{z_1}}{\bar{z_2}} \tag{3.1}$$

$$\overline{z_1 z_2} = (r_1 e^{i\theta_1} r_2 e^{i\theta_2})^* = (r_1 r_2 e^{i(\theta_1+\theta_2)})^* = r_1 r_2 e^{-i(\theta_1+\theta_2)} = r_1 e^{-i\theta_1} r_2 e^{-i\theta_2} = \bar{z_1}\bar{z_2} \tag{3.2}$$

$$\overline{z_1 - z_2} = \overline{x_1 + iy_1 - x_2 - iy_2} = x_1 - iy_1 - x_2 + iy_2 = \overline{x_1 + iy_1} - \overline{x_2 + iy_2} = \bar{z_1} - \bar{z_2} \tag{3.3}$$

4 Ch. 2, §5C, p. 53, 28

$$\left|\frac{z}{\bar{z}}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = \left|e^{2i\theta}\right| = 1 \tag{4.1}$$

5 Ch. 2, §5D, p. 54, 50

We want to find all $x, y \in \mathbb{R}$ such that $|x+iy| = y - ix$. Note that the left hand side is a positive real number, so the right hand side must also be a positive real number. Thus, $0 = Im(y - ix) = -x$ so $x = 0$. This reduces the equation to $|iy| = y$, which holds for $y \geq 0$.

6 Ch. 2, §7, p. 59, 12

$$\rho = \lim_{n\to\infty}\left|\frac{((n+1)!)^2 z^{n+1}}{(2n+2)!}\frac{(2n)!}{(n!)^2 z^n}\right| = \lim_{n\to\infty}\left|\frac{z(n+1)^2}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{z}{4}\frac{n+1}{n+1/2}\right| = |z/4| \tag{6.1}$$

The disc of convergence is when $\rho < 1$, so $|z/4| < 1 \Rightarrow |z| < 4$.

7 Ch. 2, §9, p. 64, 25

$$\left(\frac{i\sqrt{2}}{1+i}\right)^{12} = \left(\frac{\sqrt{2}e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}}\right)^{12} = \left(e^{i\pi/4}\right)^{12} = e^{i3\pi} = -1 \tag{7.1}$$

8 Ch. 2, §10, p. 67, 32

The three cube roots of 1 are $e^{i2\pi n/3}$, $n = 0, 1, 2$. $n = 0$ gives the root $e^{i0} = 1$. $n = 1$ gives the root $\omega = e^{2\pi i/3}$, and $n = 2$ gives $\omega^2 = e^{4\pi i/3}$. Also, note that $(\omega^2)^2 = e^{8\pi i/3} = e^{2\pi i/3} = \omega$.

9 Ch. 2, §12, p. 71, 27

$$\begin{aligned}
\sin(4+3i) &= \sin(4)\cos(3i) + \cos(4)\sin(3i) = \sin(4)\cosh(3) + i\cos(4)\sinh(3) && (9.1)\\
\Rightarrow \Re(\sin(4+3i)) &= \sin(4)\cosh(3) = -7.619 && (9.2)\\
\Im(\sin(4+3i)) &= \cos(4)\sinh(3) = -6.548 && (9.3)\\
|\sin(4+3i)| &= \sqrt{\sin^2(4)\cosh^2(3) + \cos^2(4)\sinh^2(3)} = 10.046 && (9.4)
\end{aligned}$$

10 Ch. 2, §14, p. 74, 24

a.) $(-i)^{(2+i)(2-i)} = (-i)^5 = -i^5 = -i$

But $(-i)^{2+i} = e^{(2+i)ln(-i)} = e^{(2+i)(\frac{3\pi i}{2}+2\pi i n_1)} = e^{\frac{6\pi i}{2}+4\pi i n_1-\frac{3\pi}{2}-2\pi n_1} = e^{\pi i-\frac{3\pi}{2}-2\pi n_1}$ where $n_1 \in \mathbb{Z}$

Thus, $[(-i)^{2+i}]^{2-i} = [e^{\pi i-\frac{3\pi}{2}-2\pi n_1}]^{2-i} = e^{(2-i)(\pi i-\frac{3\pi}{2}-2\pi n_1+2\pi i n_2)} = e^{2\pi i-3\pi-4\pi n_1+4\pi i n_2+\pi+\frac{3\pi i}{2}-2\pi i n_1+2\pi n_2}$
$= e^{-3\pi-4\pi n_1+\pi+\frac{3\pi i}{2}+2\pi n_2} = -ie^{2\pi N}$ where $N \in \mathbb{Z}$

Note that $N = 0$ gives the result of the first formula.

b.) $i^{-1} = \frac{1}{i} = -i$

But, $i^i = e^{iln(i)} = e^{i(\frac{\pi i}{2}+2\pi i n_1)} = e^{-\frac{\pi}{2}-2\pi n_1}$ where $n_1 \in \mathbb{Z}$

Thus, $[i^i]^i = [e^{-\frac{\pi}{2}-2\pi n_1}]^i = e^{i(-\frac{\pi}{2}-2\pi n_1+2\pi i n_2)} = e^{-\frac{\pi i}{2}-2\pi i n_1-2\pi n_2} = e^{-\frac{\pi i}{2}-2\pi n_2} = -ie^{-2\pi n_2}$ where $n_2 \in \mathbb{Z}$

Note that this is the same general formula as part (a) and the two sections of part (b) agree when $n_2 = 0$.

11 Ch. 2, §15, p. 76, 18

Suppose $\exists z$ such that $\tanh z = \pm 1$.

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = \pm 1 \tag{11.1}$$

$$\Rightarrow e^z - e^{-z} = \pm e^z \pm e^{-z} \tag{11.2}$$

$$\Rightarrow (1 \mp 1)e^z - (1 \pm 1)e^{-z} = 0 \tag{11.3}$$

$$\Rightarrow e^z = 0 \quad or \quad e^{-z} = 0 \tag{11.4}$$

But from eq. 11.1, $e^z = e^x(cos(y) + isin(y))$ which is never zero. Thus, $\nexists z$ such that $\tanh z = \pm 1$.