105A PRACTICE PROBLEMS

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Disclaimer

Think of these problems as a hypothetical take home final. These problems are likely more involved than what you will see on the actual final (so hopefully they will make for good practice). Also, I'm only covering material from the second half of the quarter, but on the real final anything is fair game. So make sure you review the midterm and your old homework assignments too.

Problem 1: Lagrangians and Conserved Quantities

Consider the following action for a particle of mass m moving in one dimension

$$S = \int dt\mathcal{L} = -mc^2 \int dt\sqrt{1 - \frac{\dot{x}^2}{c^2}}.$$

(1) What are the units of c? What does this make the units of S? Does this agree with what you're used to?
(2) Calculate the generalized momentum p for the particle. Write down the equation of motion for the particle. You may write the EOM in terms of p if you wish. (Hint: This will make things much simpler.) (Extra Credit: In the limit $p \to \infty$ what is the value of $\dot{x}$?)
(3) Write $\dot{x}$ as a function of p. Now write the Lagrangian $\mathcal{L}$ as a function of p.
(4) Recall that the Hamiltonian is defined as

$$\mathcal{H} = p\dot{x} - \mathcal{L}.$$

Now, using your results from the previous part, write the Hamiltonian as a function of p only. Simplify your expression. Is $\mathcal{H}$ a constant? Why or why not?
(5) Write a Taylor expansion of $\mathcal{H}$ in the limit $p \ll mc$ to first order in p/mc. Does this look familiar? (Extra Credit: Calculate $\mathcal{H}$ to second order in p/mc. This is the first relativistic correction to the energy of a moving particle. Notice that the first order term is the only term that does not involve c.)

For those of you who have studied relativity, $S = -mc^2 \int d\tau$ where τ is the proper time of the particle's path. So, in the theory of relativity (unlike in Newtonian Mechanics) the action has a very nice physical interpretation, it's just proportional to the time measured by a moving clock.

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Problem 2: Constrained Systems

The purpose of this problem is to practice using Lagrange multipliers. To do this we will analyze a problem that naturally lends iteself to polar coordinates in cartesian coordinates

Consider a particle of mass m constrained to move in a circle of radius R. The particle experience no forces (other than the constraint forces keeping it in the circle).

(1) Write down the Lagrangian $\mathcal{L}$ for the *unconstrained* system in cartesian coordinates.
(2) Now write down a constraint $f(x, y)$ such that $f(x, y) = 0$ when the particle lies on the circle.
(3) Argue that $\mathcal{L}' = \mathcal{L} + \lambda f$ is just as good of a Lagrangian as $\mathcal{L}$ was.
(4) Write down the equations of motion for x and y and write down the general solution to these equations.
(5) Find a solution that is consistent with the constraint $f(x, y) = 0$. (Hint: Don't over-think this. Remember λ was arbitrary, it could be positive or negative.)
(6) Calculate the constraint forces F_x^{cstr} and F_y^{cstr}. Calculate $|F^{cstr}| = \sqrt{(F_x^{cstr})^2 + (F_y^{cstr})^2}$. (Hint: Do you see anything that looks like a potential in $\mathcal{L}'$?)
(7) Does your solution allow for periodic motion? (If not you've made a mistake!) Write down an expression that relates the angular frequency ω to F^{cstr}.
(8) Let's say I told you that $|F^{cstr}| = k/R^2$. Plug this into your answer for the previous part and write down an expression that relates R and the period T (time to complete one orbit). Write your expression in the form $T = cR^p$ where c is some constant. What is p? Where have you seen this relationship before?

Problem 3: Oscillatory Motion

Consider a sinusoidally driven, damped harmonic oscillator, i.e. a system described by the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t).$$

You may take $m = 1$ if you wish. As you know this system has a solution of the form

$$x(t) = A\cos(\omega t + \delta)$$
$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}$$
$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right).$$

In addition there are so called "transient" solutions, which depend on initial conditions and which decay exponentially. Assume we have waited long enough so that these transients are negligible and the above solution is a very good approximation.

For all of the calculations below assume the system is driven on resonance ($\omega = \omega_0$). Also, you may find this identity helpful

$$\sin(A + B) = \cos(A)\sin(B) + \sin(A)\cos(B).$$

(1) What is the average total energy (kinetic plus potential) stored in the oscillator? To do this calculate the energy at a time t, integrate the energy over a full period $t = (0, T)$, and divide by T. (The stored energy does not include energy supplied by the external driving force).

(2) Calculate the total power supplied by the driving force per cycle. Remember $P = \mathbf{F} \cdot \mathbf{v}$.

(3) A common figure of merit for characterizing damped oscillators is the "quality factor" Q. Q is defined as

$$Q = 2\pi \frac{\text{(Average Stored Energy)}}{\text{(Energy Lost per Cycle)}},$$

when the oscillator is driven on resonance. Another way to say the same thing is to define $1/Q$ as the fraction of the energy of the oscillator that is lost with each cycle (divided by 2π).

Using your results from the previous parts calculate Q.

(4) Imagine that the driving force is abruptly turned off. Roughly how many oscillations would you expect the oscillator to complete before stopping. State your answer in terms of Q. Clarification: You probably know that the does not actually go to zero but decay exponentially. In the real world the oscillator will come to rest and the time this takes will be, to within a factor of 10 (which is what physicists usually mean when they say "roughly") the time it takes for the amplitude to decrease by one factor of e.

(5) Extra Credit: Derive the expressions for A and δ given above.