

PROBLEM 1: LAGRANGIANS AND CONSERVED QUANTITIES

Consider the following action for a particle of mass m moving in one dimension

$$S = \int dt \mathcal{L} = -mc^2 \int dt \sqrt{1 - \frac{\dot{x}^2}{c^2}}.$$

- (1) What are the units of c ? What does this make the units of S ? Does this agree with what you're used to?

c must have units of velocity in order for $1 - \dot{x}^2/c^2$ to make sense as an expression. Therefore the action has units of (mass) · (length)²/(time) (where we remembered to count dt). So, the action has units of (energy) · (time), as we expect.

- (2) Calculate the generalized momentum p for the particle. Write down the equation of motion for the particle. You may write the EOM in terms of p if you wish. (Hint: This will make things much simpler.) (Extra Credit: In the limit $p \rightarrow \infty$ what is the value of \dot{x} ?)

Using the definition of the generalized momentum we find

$$\begin{aligned} p \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} &= -mc^2 \frac{1}{2} \frac{1}{\sqrt{1 - \dot{x}^2/c^2}} \frac{-2\dot{x}}{c^2} \\ &= \frac{m\dot{x}}{\sqrt{1 - \dot{x}^2/c^2}}. \end{aligned}$$

Since \mathcal{L} has no explicit dependence on x the Euler-Lagrange equation is simply

$$\frac{dp}{dt} = 0.$$

Finally we see that in the limit $p \rightarrow \infty$, $\dot{x} \rightarrow c$.

- (3) Write \dot{x} as a function of p . Now write the Lagrangian \mathcal{L} as a function of p .

Doing a little algebra we find that

$$\dot{x} = \frac{p/m}{\sqrt{1 + (p/mc)^2}}$$

and

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} = -\frac{mc^2}{\sqrt{1 + (p/mc)^2}}$$

- (4) Recall that the Hamiltonian is defined as

$$\mathcal{H} = p\dot{x} - \mathcal{L}.$$

Now, using your results from the previous part, write the Hamiltonian as a function of p only. Simplify your expression. Is \mathcal{H} a constant? Why or why not?

Using our previous results we find

$$\begin{aligned} \mathcal{H} &= p \frac{p/m}{\sqrt{1 + (p/mc)^2}} - \left(-\frac{mc^2}{\sqrt{1 + (p/mc)^2}} \right) \\ &= mc^2 \sqrt{1 + (p/mc)^2} \end{aligned}$$

\mathcal{H} is a constant for two reasons. First, \mathcal{H} is a function of p only and the Euler-Lagrange equation tells us the p is constant. More generally you proved on your homework that \mathcal{H} is constant whenever the Lagrangian has no explicit time dependence. Since \mathcal{L} depends only on \dot{x} , \mathcal{H} must be conserved.

- (5) Write a Taylor expansion of \mathcal{H} in the limit $p \ll mc$ to first order in p/mc . Does this look familiar? (Extra Credit: Calculate \mathcal{H} to second order in p/mc . This is the first relativistic correction to the energy of a moving particle. Notice that the first order term is the only term that does not involve c .)

Using the general result $(1 + \epsilon)^n \approx 1 + n\epsilon + n(n-1)\epsilon^2/2 + \dots$ we find

$$\mathcal{H} \approx mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots$$

Since in this limit we also have $p \approx m\dot{x}$, the lowest order term (neglecting the constant mc^2) gives us the familiar expression for Kinetic energy.

For those of you who have studied relativity, $S = -mc^2 \int d\tau$ where τ is the proper time of the particle's path. So, in the theory of relativity (unlike in Newtonian Mechanics) the action has a very nice physical interpretation, it's

just proportional to the time measured by a moving clock.

This is easily seen using the Lorentz invariant

$$\begin{aligned}c^2 d\tau^2 &= c^2 dt^2 - dx^2 \\ &= \left[c^2 - \left(\frac{dx}{dt} \right)^2 \right] dt^2 \\ d\tau &= \sqrt{1 - \dot{x}^2/c^2} dt.\end{aligned}$$

PROBLEM 2: CONSTRAINED SYSTEMS

The purpose of this problem is to practice using Lagrange multipliers. To do this we will analyze a problem that naturally lends itself to polar coordinates in cartesian coordinates

Consider a particle of mass m constrained to move in a circle of radius R . The particle experience no forces (other than the constraint forces keeping it in the circle).

- (1) Write down the Lagrangian \mathcal{L} for the *unconstrained* system in cartesian coordinates.

$$\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2)$$

- (2) Now write down a constraint $f(x, y)$ such that $f(x, y) = 0$ when the particle lies on the circle.

$$f(x, y) = x^2 + y^2 - R^2$$

- (3) Argue that $\mathcal{L}' = \mathcal{L} + \lambda f$ is just as good of a Lagrangian as \mathcal{L} was.

At the end of the day we are only interested in taking solution for which $f(x, y) = 0$, so on an actual solution we have $\mathcal{L} = \mathcal{L}'$.

- (4) Write down the equations of motion for x and y and write down the general solution to these equations.

The Euler-Lagrange equations of \mathcal{L}' are

$$\begin{aligned} m\ddot{x} &= 2\lambda x \\ m\ddot{y} &= 2\lambda y. \end{aligned}$$

Without making up our mind about the sign of λ we can write

$$\begin{aligned} x(t) &= A \exp \left[\left(\sqrt{2\lambda/m} \right) t \right] + B \exp \left[- \left(\sqrt{2\lambda/m} \right) t \right] \\ y(t) &= C \exp \left[\left(\sqrt{2\lambda/m} \right) t \right] + D \exp \left[- \left(\sqrt{2\lambda/m} \right) t \right]. \end{aligned}$$

- (5) Find a solution that is consistent with the constraint $f(x, y) = 0$. (Hint: Don't over-think this. Remember λ was arbitrary, it could be positive or negative.)

We could plug our solutions plus appropriate initial data into the constraint and eliminate all but one of the integration constants above, however, it's not hard to guess that the solution we are looking for is

$$\begin{aligned}x(t) &= R \cos(\omega t + \delta) \\y(t) &= R \sin(\omega t + \delta).\end{aligned}$$

This solution clearly satisfies the constraints, and is a solution to the Euler-Lagrange equations when $\lambda < 0$ and

$$\omega = \sqrt{\frac{2|\lambda|}{m}}.$$

- (6) Calculate the constraint forces F_x^{cstr} and F_y^{cstr} . Calculate $|F^{cstr}| = \sqrt{(F_x^{cstr})^2 + (F_y^{cstr})^2}$. (Hint: Do you see anything that looks like a potential in \mathcal{L}' ?)

The only potential-like term in \mathcal{L}' comes from the constraint. So the constraint force is $\lambda \nabla f(x, y) = \nabla \mathcal{L}'$. In other words, since there are no non-constraint forces, the constraint forces are simply equal to the generalized forces

$$\begin{aligned}F_x^{cstr} &= \frac{\partial \mathcal{L}}{\partial x} = 2\lambda x \\F_y^{cstr} &= \frac{\partial \mathcal{L}}{\partial y} = 2\lambda y \\|F^{cstr}| &= 2|\lambda|R.\end{aligned}$$

- (7) Does your solution allow for periodic motion? (If not you've made a mistake!) Write down an expression that relates the angular frequency ω to F^{cstr} .

Yes, and we see that

$$|F^{cstr}| = m\omega^2 R.$$

As we know from other calculations, this is precisely the force needed to constrain a particle to move in a circle, so everything checks out.

- (8) Let's say I told you that $|F^{cstr}| = k/R^2$. Plug this into your answer for the previous part and write down an expression that relates R and the period T (time to complete one orbit). Write your expression in the form $T = cR^p$ where c is some constant. What is p ? Where have you seen this relationship before?

Making the substitution $\omega = 2\pi/T$ we can write

$$\frac{k}{R^2} = m \left(\frac{2\pi}{T} \right)^2 R$$
$$T = 2\pi \sqrt{\frac{m}{k}} R^{3/2}.$$

So, $p = 3/2$ and we have recovered Kepler's third law of planetary motion. This is equivalent to the first calculation that Newton did to show that Kepler's laws support a $1/R^2$ force law for gravity.

PROBLEM 3: OSCILLATORY MOTION

Consider a sinusoidally driven, damped harmonic oscillator, i.e. a system described by the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t).$$

You may take $m = 1$ if you wish. As you know this system has a solution of the form

$$\begin{aligned} x(t) &= A \cos(\omega t + \delta) \\ A^2 &= \frac{f_0^2}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2} \\ \delta &= \tan^{-1} \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right). \end{aligned}$$

In addition there are so called “transient” solutions, which depend on initial conditions and which decay exponentially. Assume we have waited long enough so that these transients are negligible and the above solution is a very good approximation.

For all of the calculations below assume the system is driven on resonance ($\omega = \omega_0$). Also, you may find this identity helpful

$$\sin(A + B) = \cos(A) \sin(B) + \sin(A) \cos(B).$$

- (1) What is the average total energy (kinetic plus potential) stored in the oscillator? To do this calculate the energy at a time t , integrate the energy over a full period $t = (0, T)$, and divide by T . (The stored energy does not include energy supplied by the external driving force).

The expressions above simplify when $\omega = \omega_0$ and we can write

$$\begin{aligned} \langle E \rangle &= \frac{1}{T} \int_0^T dt \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega_0^2 x^2 \right) \\ &= \frac{A^2 \omega_0^2}{2T} \int_0^T dt (\sin^2(\omega_0 t + \delta) + \cos^2(\omega_0 t + \delta)) \\ &= \frac{A^2 \omega_0^2}{2} = \frac{f_0^2}{8\beta^2}. \end{aligned}$$

- (2) Calculate the average power supplied by the driving force. Remember $P = \mathbf{F} \cdot \mathbf{v}$.

Similarly

$$\begin{aligned}
 \langle P \rangle &= \frac{1}{T} \int_0^T dt [f_0 \cos(\omega_0 t)] [-A\omega_0 \sin(\omega_0 t + \delta)] \\
 &= -\frac{f_0 A \omega_0}{T} \int_0^T dt \cos(\omega_0 t) [\sin(\omega_0 t) \cos(\delta) + \cos(\omega_0 t) \sin(\delta)] \\
 &= -\frac{f_0 A \omega_0 \sin(\delta)}{T} \int_0^T dt \cos^2(\omega_0 t) \\
 &= -\frac{f_0 A \omega_0 \sin(\delta)}{2} = -\frac{f_0^2}{4\beta}
 \end{aligned}$$

- (3) A common figure of merit for characterizing damped oscillators is the “quality factor” Q . Q is defined as

$$Q = 2\pi \frac{(\text{Average Stored Energy})}{(\text{Energy Lost per Cycle})},$$

when the oscillator is driven on resonance. Another way to say the same thing is to define $1/Q$ as the fraction of the energy of the oscillator that is lost with each cycle (divided by 2π).

Using your results from the previous parts calculate Q .

Since the average stored energy is constant, the energy lost per cycle must be exactly cancel the energy put in by the driving force. So, the energy lost per cycle must be $-\langle P \rangle T$. Therefore,

$$\begin{aligned}
 Q &= 2\pi \frac{\langle E \rangle}{-\langle P \rangle T} = \left(\frac{2\pi}{T} \right) \frac{f_0^2/8\beta^2}{f_0^2/4\beta} \\
 &= \frac{\omega_0}{2\beta}.
 \end{aligned}$$

So we see that the magnitude of the driving force f_0 has nicely canceled out and we are left with a quantity that characterizes the oscillator.

- (4) Imagine that the driving force is abruptly turned off. Roughly how many oscillations would you expect the oscillator to complete before stopping. State your answer in terms of Q . Clarification: You probably know that the does not actually go to zero but decay exponentially. In the real world the oscillator will come to rest and the time this takes will be, to within a factor of 10 (which is what physicists usually mean when they say “roughly”) the time it takes for the amplitude to decrease by one factor of e .

By the definition of Q

$$\Delta E = -2\pi \frac{\langle E \rangle}{Q}.$$

Taking $\langle E \rangle$ to be an instantaneous function of time $E(t)$ we can write the approximate equation (good for large Q)

$$\frac{dE}{dt} T \approx -2\pi \frac{E}{Q},$$

which has the solution

$$E(t) = E_0 \exp\left(-\frac{2\pi t}{QT}\right).$$

So the time it takes for the energy to decrease by a factor of e is $QT/2\pi$ or $Q/2\pi$ cycles. The energy is proportional to x^2 so the amplitude falls of by a factor of e in roughly half the time, or after roughly Q/π cycles.

- (5) Extra Credit: Derive the expressions for A and δ given above.

See pages 181-183 of Taylor.