UNIVERSITY OF CALIFORNIA, SANTA BARBARA

Department of Physics

Physics 105A — Winter 2012

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## Midterm Exam

This exam is 50 minutes long. You may refer only to notes you have brought with you on one 8 1/2 x 11 sheet of paper. Each problem is worth 10 points.

1. A particle subject to linear air resistance is thrown vertically down with a speed $v_0$ which is twice the terminal velocity. Compute how the velocity varies with time.

$\downarrow y$

Letting $y$ decrease downward:

$$m\ddot{y} = mg - bv$$

$$\Rightarrow \quad \dot{v} = g - \frac{b}{m}v \quad \Rightarrow \quad \boxed{v_{ter} = \frac{mg}{b}}$$

Let $\tilde{v} = v - \frac{mg}{b} = v - v_{ter}$

Then $\dot{\tilde{v}} = -\frac{b}{m}\tilde{v} \Rightarrow \tilde{v} = Ae^{-bt/m}$

So $v(t) = Ae^{-bt/m} + v_{ter}$

Initial condition: $v(0) = 2v_{ter} \Rightarrow A = v_{ter}$

$$\Rightarrow \boxed{v(t) = v_{ter}\left(1 + e^{-bt/m}\right)}$$

The particle slows down with time, even though gravity is trying to accelerate it.

2. A rocket with initial mass $m_0$ is accelerating from rest in empty space. At first, as it speeds up, its momentum $p$ increases. But later, as its mass $m$ decreases, $p$ eventually starts to decrease. For what value of $m$ is the momentum a maximum?

The rocket equation is

$$m \frac{dv}{dt} = -\frac{dm}{dt} v_{ex} \qquad (*)$$

where $v_{ex}$ is the exhaust velocity.

$$p = mv \Rightarrow \frac{dp}{dt} = \dot{m} v + \underbrace{m\dot{v}}_{-\dot{m} v_{ex}}$$

$$= \dot{m}(v - v_{ex})$$

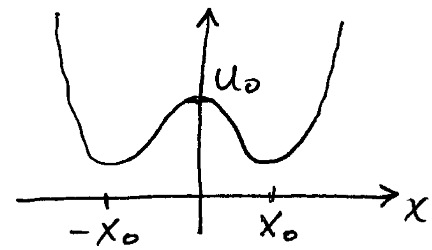
So $p$ is a maximum when $v = v_{ex}$.

What is $m$ when $v = v_{ex}$? From $(*)$

$$\frac{dv}{v_{ex}} = -\frac{dm}{m} \Rightarrow \frac{v}{v_{ex}} = -\ln(m/m_0)$$

So $v = v_{ex} \Rightarrow \ln(m/m_0) = -1 \Rightarrow \boxed{m = m_0/e}$

3. A particle of mass $m$ is moving in a potential




$$U(x) = U_0 - \frac{1}{2}\alpha x^2 + \frac{1}{4}\lambda x^4$$

where $\alpha$ and $\lambda$ are positive constants.

a) Find the points of equilibrium and state whether they are stable, unstable, or marginal.

b) For the stable ones, what is the angular frequency for small oscillations?

a) $U'(x) = -\alpha x + \lambda x^3$

Equilibria are points with $U'(x) = 0$. So

$\boxed{x = 0}$ or $\lambda x^2 = \alpha \Rightarrow \boxed{x = x_0 = \pm\sqrt{\alpha/\lambda}}$

$U''(x) = -\alpha + 3\lambda x^2$, $U''(x_0) = 3\alpha > 0$

So $\boxed{\begin{array}{l} x = 0 \text{ is unstable,} \\ x = \pm\sqrt{\alpha/\lambda} \text{ are stable} \end{array}}$

b) Expand $U(x)$ about $x = x_0$

$$U(x) = U(x_0) + \tfrac{1}{2} U''(x_0)(x - x_0)^2 + \cdots$$

This is like a spring with spring constant $k = U''(x_0)$. So $\omega^2 = k/m = \dfrac{U''(x_0)}{m}$

$$\boxed{\omega = \sqrt{\frac{3\alpha}{m}}}$$