Phys 105A  Solutions #4

We first calculate $I$:

If the density is $\rho$, the mass is

$$M = \frac{4}{3} \pi R^3 \rho$$

The moment of inertia is

$$I = \int (r \sin \theta)^2 \rho \, dV$$

$$= \rho \int_0^R r^4 \, dr \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r \sin \theta)^2 \sin \theta \, d\phi \, d\theta$$

$$= \rho \int_0^R r^4 \, dr \left[ \int_0^{\frac{\pi}{2}} \sin^3 \theta \, d\theta \right]$$

and

$$\int_0^{\frac{\pi}{2}} \sin^3 \theta \, d\theta = \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \sin \theta \, d\theta$$

$$= -\cos \theta \bigg|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\cos \theta$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

So

$$I = \frac{8\pi}{15} R^5 \rho = \frac{2}{5} MR^2$$

So the angular momentum is $\vec{L} = \vec{I}\vec{\omega}$

or $L_2 = I\omega$ (assuming rotation about z-axis).
(2) A force is conservative if

\[(\vec{\nabla} \times \vec{F})_x = \partial_y F_z - \partial_z F_y = 0\]

\[(\vec{\nabla} \times \vec{F})_y = \partial_z F_x - \partial_x F_z = 0\]

\[(\vec{\nabla} \times \vec{F})_z = \partial_x F_y - \partial_y F_x = 0\]

(a) If \( \vec{F} = (ax, by^2, cz^3) \), each term on the rhs vanishes, so the force is conservative. The potential is found by solving \( \vec{F} = -\vec{\nabla} U \):

\[ U = -\left(\frac{a}{2}x^2 + \frac{b}{3}y^3 + \frac{c}{4}z^4\right) \]

CK: \( \frac{\partial U}{\partial x} = -ax = -F_x \) etc.

(b) \( (\vec{\nabla} \times \vec{F})_z = \partial_x F_y = b \neq 0 \). Force is not conservative.

(c) \( (\vec{\nabla} \times \vec{F})_z = a - a = 0 \), \( (\vec{\nabla} \times \vec{F})_x = 0 \), \( (\vec{\nabla} \times \vec{F})_y = 0 \)

So the force is conservative, and

\[ U = -axy \]

Since \( \frac{\partial U}{\partial x} = -ay = -F_x \); \( \frac{\partial U}{\partial y} = -ax = -F_y \)

\( \frac{\partial U}{\partial z} = 0 = -F_z \)
(4.2) (3) \( \mathbf{F} = (x^2, 2xy) \) Compute \( W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} \)

(a) \( W = \int_0^1 (\mathbf{F} \cdot \mathbf{x}) \, dx + \int_0^1 (\mathbf{F} \cdot \mathbf{y}) \, dy \)

\[
W = \int_0^1 x^2 \, dx + \int_0^1 2y \, dy = \frac{1}{3} + 1 = \frac{4}{3}
\]

(b) Path \( y = x^2 \) \( \Rightarrow \) \( dy = 2x \, dx \)

So \( W = \int F_x \, dx + \int F_y \, dy \)

\[
= \int_0^1 x^2 \, dx + \int_0^1 2x \cdot x^2 (2x \, dx) = \frac{1}{3} + \frac{4}{5} = \frac{17}{15}
\]

(c) \( x = t^3, y = t^2 \) \( \Rightarrow \) \( dx = 3t^2 \, dt, dy = 2t \, dt \)

So \( W = \int F_x \, dx + \int F_y \, dy \)

\[
= \int_0^1 t^6 (3t^2 \, dt) + \int_0^1 2t^3 \cdot t^2 (2t \, dt) = 3 \int_0^1 t^8 \, dt + 4 \int_0^1 t^6 \, dt = \frac{1}{3} + \frac{4}{7} = \frac{19}{21}
\]

Note that since \( F_x \) is independent of \( y \), its contribution to \( W \) is independent of the path.
Initially the puck is at height $h = 2R + v \approx 0$. So
$T = 0$, $U = mgh = 2mgR$

The total energy is $E = 2mgR$

At any later time:
$\frac{1}{2}mv^2 + mgh = 2mgR$

Since energy is conserved. So $v^2 = 2g(2R-h)$

If the puck is at an angle $\phi$ from the vertical, the normal force due to gravity is:
$mg \cos \phi = mg \frac{(h-R)}{R}$

The puck leaves the sphere when this equals the centripetal acceleration $v^2/R$.

Acceleration due to gravity:
$g \frac{(h-R)}{R} = \frac{v^2}{R} = \frac{2g(2R-h)}{R}$

Solve for height:
$h = \frac{5}{3}R$
(4.28) (5) \[ U = \frac{1}{2} K x^2 \]

(a) \[ E = T + U = \frac{1}{2} m v^2 + \frac{1}{2} K x^2 \]

So \[ v = \pm \sqrt{\frac{1}{m} (E - \frac{1}{2} K x^2)} \]

\[ \dot{x} = v = \pm \sqrt{\frac{1}{m} (2E - K x^2)} \] (1)

(b) Initially, \( x = A \) and \( v = 0 \) at maximum displacement, so \( E = \frac{1}{2} K A^2 \). But its conserved. Sub into (1):

\[ \dot{x} = \pm \sqrt{\frac{K}{m} (A^2 - x^2)} \]

So

\[ t = \int_0^x \frac{dx'}{\dot{x}} = \sqrt{\frac{m}{K}} \int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}} \]

Let \( x' = A \sin \theta \):

\[ t = \sqrt{\frac{m}{K}} \int d\theta = \sqrt{\frac{m}{K}} \theta \]

So

\[ t = \sqrt{\frac{m}{K}} \arcsin \frac{x}{A} \]

(c) \( x(t) = A \sin \sqrt{\frac{K}{m}} t \)

So mass executes simple harmonic motion with period \( 2\pi \sqrt{\frac{m}{K}} \)