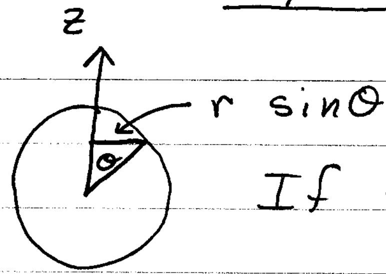
Phys 105A Solutions #4

(1) 

We first calculate I:
If the density is ρ, the mass is
$$M = \frac{4}{3}\pi R^3 \rho$$

The moment of inertia is

$$I = \int (r\sin\theta)^2 \rho \, dV$$
$$= \rho \int_0^R dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, (r\sin\theta)^2 (r^2 \sin\theta)$$
$$= \underbrace{\rho \int_0^R r^4 \, dr}_{\frac{R^5}{5}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi} \sin^3\theta \, d\theta$$

and
$$\int_0^{\pi} \sin^3\theta \, d\theta = \int_0^{\pi} (1-\cos^2\theta)\sin\theta \, d\theta$$
$$= -\cos\theta \Big|_0^{\pi} + \int_{+1}^{-1} \cos^2\theta \, d(\cos\theta)$$
$$= 2 - \frac{2}{3} = \frac{4}{3}$$

So
$$I = \frac{8\pi}{15} R^5 \rho = \frac{2}{5} M R^2$$

So the angular momentum is $\vec{L} = I\vec{\omega}$
or $L_z = I\omega$ (assuming rotation about z-axis).

(2) A force is conservative if

$$(\vec{\nabla}\times\vec{F})_x = \partial_y F_z - \partial_z F_y = 0$$

$$(\vec{\nabla}\times\vec{F})_y = \partial_z F_x - \partial_x F_z = 0$$

$$(\vec{\nabla}\times\vec{F})_z = \partial_x F_y - \partial_y F_x = 0$$

(a) If $\vec{F} = (ax, by^2, cz^3)$, each term on the rhs vanishes, so the force is conservative. The potential is found by solving $\vec{F} = -\vec{\nabla} U$:

$$\boxed{U = -\left(\frac{a}{2}x^2 + \frac{b}{3}y^3 + \frac{c}{4}z^4\right)}$$

CK: $\frac{\partial U}{\partial x} = -ax = -F_x$ etc.

(b) $(\vec{\nabla}\times\vec{F})_z = \partial_x F_y = b \neq 0$. Force is <u>not</u> conservative.

(c) $(\vec{\nabla}\times\vec{F})_z = a - a = 0$, $(\vec{\nabla}\times\vec{F})_x = 0$, $(\vec{\nabla}\times\vec{F})_y = 0$

So the force is conservative, and

$$\boxed{U = -axy}$$

since $\frac{\partial U}{\partial x} = -ay = -F_x$; $\frac{\partial U}{\partial y} = -ax = -F_y$

$$\frac{\partial U}{\partial z} = 0 = -F_z$$

(4.2) (3) $\vec{F} = (x^2, 2xy)$ Compute $W = \int_0^P \vec{F}\cdot d\vec{r}$

(a) $W = \int_0^1 (\vec{F}\cdot\hat{x})dx + \int_0^1 (\vec{F}\cdot\hat{y})\,dy$
$\quad (y=0) \qquad (x=1)$

$$W = \int_0^1 x^2 dx + \int_0^1 2y\,dy = \frac{1}{3} + 1 = \boxed{\frac{4}{3}}$$

(b) Path $y = x^2 \Rightarrow dy = 2x\,dx$

So $W = \int F_x\,dx + \int F_y\,dy$

$$= \int_0^1 x^2 dx + \int_0^1 2x\cdot x^2 (2x\,dx)$$

$$= \int_0^1 (x^2 + 4x^4)\,dx = \frac{1}{3} + \frac{4}{5} = \boxed{\frac{17}{15}}$$

(c) $x = t^3,\ y = t^2 \Rightarrow dx = 3t^2 dt,\ dy = 2t\,dt$

So $W = \int F_x\,dx + \int F_y\,dy$

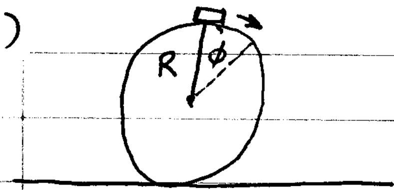
$$= \int_0^1 t^6(3t^2 dt) + \int_0^1 2\,t^3\cdot t^2(2t\,dt)$$

$$= 3\int_0^1 t^8 dt + 4\int_0^1 t^6 dt$$

$$= \frac{1}{3} + \frac{4}{7} = \boxed{\frac{19}{21}}$$

Note that since F_x is independent of y, its contribution to W is independent of the path.

(4.8) (4)



Initially the puck is at height $h = 2R$ & $v \approx 0$. So

$$T = 0, \quad U = mgh = 2mgR$$

The total energy is $E = 2mgR$

At any later time:

$$\frac{1}{2}mv^2 + mgh = 2mgR$$

since energy is conserved. So $\boxed{v^2 = 2g(2R-h)}$

If the puck is at an angle ϕ from the vertical, the normal force due to gravity is $mg\cos\phi = mg\frac{(h-R)}{R}$.

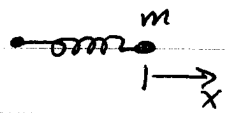
The puck leaves the sphere when this ~~equals~~ can no longer provide the centripetal acceleration v^2/R.

Acceleration due to gravity: $g\frac{(h-R)}{R} = \frac{v^2}{R} = \frac{2g(2R-h)}{R}$

Solve for height:

$$\boxed{h = \frac{5}{3}R}$$

(4.28) (5)



$$U = \tfrac{1}{2} k x^2$$

(a) $E = T + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

So $v = \pm \sqrt{\left(E - \frac{1}{2} k x^2\right)\frac{2}{m}}$

$$\dot{x} = v = \pm \sqrt{\frac{1}{m}\left(2E - k x^2\right)} \qquad (1)$$

(b) ~~Initially~~, $x = A$ and $v = 0$ at maximum displacement, so $E = \frac{1}{2} k A^2$ then. But its conserved. Sub into (1):

$$\dot{x} = \pm \sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$

So

$$t = \int_0^x \frac{dx'}{\dot{x}} = \sqrt{\frac{m}{k}} \int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}}$$

Let $x' = A \sin\theta$:

$$t = \sqrt{\frac{m}{k}} \int d\theta = \sqrt{\frac{m}{k}}\, \theta$$

So

$$\boxed{t = \sqrt{\frac{m}{k}} \arcsin \frac{x}{A}}$$

(c) $x(t) = A \sin \sqrt{\frac{k}{m}}\, t$

So mass executes simple harmonic motion with period $2\pi \sqrt{\frac{m}{k}}$