Phys 105A Solutions # 6

1. (a) We need to solve:
\[ \ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t \quad (*) \]

The general solution to the homogeneous eq. is
\[ x_h(t) = A \cos (\omega_0 t - \delta) \]

To find a solution to the inhomogeneous eq. we try (note similarity to critically damped case)
\[ x(t) = B t^p \sin \omega_0 t \]

Sub into (*) * find that this is a solution if

\[ p = 1, \quad B = \frac{F_0}{2m\omega_0} \quad \text{Complete soln is} \]

\[ x(t) = A \cos (\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin \omega_0 t \]

(b) Initial cond:
\[ x(0) = 0 \Rightarrow \delta = \frac{\pi}{2}, \quad \dot{x}(0) = 0 \Rightarrow A = 0 \]

So the soln is just the second term:
\[ x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) \]

(c) At the breaking point \( |F| = k |x_b| = 5F_0 \). This is reached when

\[ k \frac{F_0}{2m\omega_0} |t \sin(\omega_0 t)| = 5F_0 \]

or \[ |t_b \sin \omega_0 t_b| = \frac{10m\omega_0}{k} = \frac{10}{\omega_0} \]
(5.43) 2. a) If four 80 kg men cause the four springs to compress 2 cm, the weight on each spring is

\[(80 \text{ kg})(9.8 \text{ m/sec}^2) = 784 \text{ N} = Kx\]

Setting \(x = 0.02 \text{ m}\) yields \(K = 3.92 \times 10^4 \text{ N/m}\)

\(\approx 4 \times 10^4 \text{ N/m}\)

b) For two springs in parallel, the net spring constant is double: \(K_{\text{net}} = 8 \times 10^4 \text{ N/m}\)

So the frequency (not angular frequency) is

\[f = \frac{1}{2\pi} \sqrt{\frac{K_{\text{net}}}{m}}\]

\[= \frac{1}{2\pi} \sqrt{\frac{8 \times 10^4}{50}} = \frac{40}{2\pi} \approx 6.3 \text{ Hz} \approx 6 \text{ Hz}\]

c) Resonance requires the time between bumps on road equal the period of oscillation of spring

Period \(T = \frac{1}{f} = 0.16 \text{ sec}\)

Car needs to travel 80 cm apart in this time.

So speed is \(v = \frac{d}{T} = \frac{0.80 \text{ m}}{0.16 \text{ sec}} = 5 \text{ m/sec}\)
3. Use

\[
\begin{align*}
\cos \theta \cos \phi &= \frac{1}{2} [\cos (\theta + \phi) + \cos (\theta - \phi)] \\
\sin \theta \sin \phi &= \frac{1}{2} [\cos (\theta - \phi) - \cos (\theta + \phi)] \\
\sin \theta \cos \phi &= \frac{1}{2} [\sin (\theta + \phi) + \sin (\theta - \phi)]
\end{align*}
\]

So if \( m \neq n \):

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos (n \omega t) \cos (m \omega t) \, dt
\]

\[
= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos ((n+m) \omega t) + \cos ((n-m) \omega t)] \, dt
\]

\[
= \frac{\sin ((n+m) \omega t)}{2(n+m) \omega} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin ((n-m) \omega t)}{2(n-m) \omega} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\]

But \( \omega \tau = 2\pi \), so all terms vanish.

If \( m = n \neq 0 \):

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 (n \omega t) \, dt = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos (2n \omega t) + 1] \, dt
\]

The 1st integral again vanishes, but the second is \( \frac{\pi}{2} \).

The same result holds for \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin (n \omega t) \sin (m \omega t) \, dt \) since sign between the two cosines didn't matter.

Finally:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin (m \omega t) \cos (n \omega t) \, dt = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin ((n+m) \omega t + \sin (m-n) \omega t)
\]

\[
= \frac{\sin ((n+m) \omega t)}{2(n+m) \omega} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{\sin ((m-n) \omega t)}{2(m-n) \omega} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0
\]

since \( \cos (\theta) \) is an even function: \( \cos (-\theta) = \cos \theta \).
In spherical coordinates
\[ ds^2 = dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

For the curve \( r = R, \ \phi = \phi(\theta) \) we have
\[ ds^2 = R^2 \left( 1 + \sin^2 \theta \phi' \right) d\theta^2 \]
So the length is
\[ L = R \int_0^\theta \sqrt{1 + \sin^2 \theta \phi' \left( \phi' \right)^2} \ d\theta \]

Let the hemisphere have radius \( R \). If \( d \) is the distance traveled by light ray, the time is \( t = \frac{d}{c} \) (\( c \) = speed of light). Need to compute \( d \):

Angle \( AOP \) is \( 90 - \theta \), so \( R \sin \frac{\theta}{2} (90 - \theta) \) is half the distance \( AP \). So \( d(AP) = 2R \sin \frac{\theta}{2} (90 - \theta) \)

Angle \( BOP \) is \( 90 + \theta \), so \( d(PB) = 2R \sin \frac{\theta}{2} (90 + \theta) \)

The total distance is
\[ d = 2R \left[ \sin \frac{90-\theta}{2} + \sin \frac{90+\theta}{2} \right] \]

Find extremum:
\[ \frac{dt}{d\theta} = \frac{R}{c} \left( - \cos \frac{90-\theta}{2} + \cos \frac{90+\theta}{2} \right) = 0 \Rightarrow \theta = 0 \]

Is it max or min?
\[ \frac{d^2t}{d\theta^2} \bigg|_{\theta=0} = \frac{R}{2c} \left( - \sin 45 - \sin 45 \right) < 0 \]
So time is maximized.