Phys 105A Solutions # 7

(6.11) 1. $f = \sqrt{x}\sqrt{1+y'^2}$. The Euler-Lagrange equation is $\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$

f is independent of y, so $\frac{d}{dx}\left(\frac{\sqrt{x}\,y'}{\sqrt{1+y'^2}}\right) = 0$

Thus $\frac{\sqrt{x}\,y'}{\sqrt{1+y'^2}} = \sqrt{A}$ (const) $\Rightarrow x y'^2 = A(1+y'^2)$

$$\Rightarrow (x-A)\,y'^2 = A$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{A}{x-A}$$

$$\Rightarrow dy = \frac{\sqrt{A}}{\sqrt{x-A}}\,dx$$

Integrate $\Rightarrow$ $y = 2\sqrt{A(x-A)} + B$

$$\Rightarrow \frac{(y-B)^2}{4A} = x - A$$

This is the eq for a parabola.

(6.20) 2. If $f = f(y, y')$, then $\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''$ by chain rule.

But the Euler-Lagrange eq is

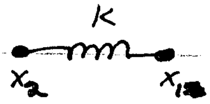
$$\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y'}$$

So

$$\frac{df}{dx} = \left(\frac{d}{dx} \frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y''$$

$$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'} y'\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'} y'\right) = 0 \Rightarrow f - \frac{\partial f}{\partial y'} y' = \text{const}$$

(7.8) 3. (a) 

The kinetic energy is

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

and the potential energy is $U = \frac{1}{2} k x^2$

so the Lagrangian is:

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k (x_1 - x_2 - \ell)^2$$

(b) Letting $X = \frac{1}{2}(x_1 + x_2)$, $x = (x_1 - x_2 - \ell)$

the Lagrangian becomes:

$$\mathcal{L}(X, x, \dot{X}, \dot{x}) = m\left(\dot{X}^2 + \frac{1}{4}\dot{x}^2\right) - \frac{1}{2} k x^2$$

The equations of motion are

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = 2m\ddot{X} \qquad \text{since } \frac{\partial \mathcal{L}}{\partial X} = 0$$

$$-kx - \frac{d}{dt}\left(m\dot{x}/2\right) = 0 \Rightarrow m\ddot{x} = -2kx$$

(c) The center of mass moves like a free particle

$$\ddot{X} = 0 \Rightarrow X(t) = vt + X_0$$

The relative position, or extension x oscillates like a simple harmonic oscillator with frequency $\omega_0^2 = \sqrt{\frac{2k}{m}}$

$$x(t) = A\cos(\omega_0 t + \delta)$$

(7.20) 4. In cylindrical coord (r, ϕ, z), the bead's position is $\vec{r} = (R, z/\lambda, z)$

It's velocity is $\vec{v} = (0, \dot{z}/\lambda, \dot{z})$

So

$$v^2 = R^2 \dot{\phi}^2 + \dot{z}^2 = \left(1 + R^2/\lambda^2\right) \dot{z}^2$$

The kinetic energy is $T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{z}^2 \left(1 + R^2/\lambda^2\right)$

The potential energy is $U = mgz$

So the Lagrangian is:

$$\boxed{\mathcal{L} = \frac{1}{2} m \dot{z}^2 \left(1 + R^2/\lambda^2\right) - mgz}$$

Euler-Lagrange eq: $\frac{d}{dt}\left[m \dot{z} \left(1 + \frac{R^2}{\lambda^2}\right)\right] + mg = 0$

$$\Rightarrow \boxed{\ddot{z} = \frac{-g}{1 + R^2/\lambda^2}}$$

When $R \to 0$, you recover the standard result $\ddot{z} = -g$ for a mass falling straight down.

(7.23) 5. The potential energy is $U = \frac{1}{2} k x^2$

The position of the small cart is

$$x(t) + X(t) = x(t) + A\cos\omega t$$

So its velocity is

$$v = \dot{x} - A\omega \sin \omega t$$

and its kinetic energy is

$$T = \frac{1}{2} m (\dot{x} - A\omega \sin\omega t)^2$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m (\dot{x} - A\omega \sin \omega t)^2 - \frac{1}{2} k x^2$$

To get the equation of motion, we use

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m(\dot{x} - A\omega \sin\omega t)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx$$

So $m\frac{d}{dt}(\dot{x} - A\omega \sin\omega t) + kx = 0$

$$m\ddot{x} - Am\omega^2 \cos\omega t + kx = 0$$

$$\Rightarrow \quad \ddot{x} + \omega_0^2 x = B\cos\omega t$$

where $\omega_0^2 = \frac{k}{m}$ and $B = A\omega^2$