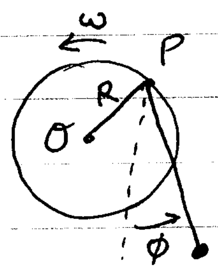
Phys 105A Solutions #8

(7.29) 1.



The position of P with respect to the origin O is $(R\cos\omega t, R\sin\omega t)$. The position of the mass with respect to P is $(l\sin\phi, -l\cos\phi)$

So the position of the mass with respect to the origin is

$$\vec{r}(t) = (l\sin\phi + R\cos\omega t, -l\cos\phi + R\sin\omega t)$$

The velocity is :

$$\vec{v} = \dot{\vec{r}} = (l\dot{\phi}\cos\phi - R\omega\sin\omega t, l\dot{\phi}\sin\phi + R\omega\cos\omega t)$$

so

$$v^2 = (l\dot{\phi}\cos\phi - R\omega\sin\omega t)^2 + (l\dot{\phi}\sin\phi + R\omega\cos\omega t)^2$$

$$= l^2\dot{\phi}^2 + R^2\omega^2 + 2\dot{\phi}lR\omega(\sin\phi\cos\omega t - \cos\phi\sin\omega t)$$

$$v^2 = l^2\dot{\phi}^2 + R^2\omega^2 + 2lR\omega\dot{\phi}\sin(\phi - \omega t)$$

The kinetic energy is $\frac{1}{2}mv^2$ and the potential energy is $U = mgy = mg(R\sin\omega t - l\cos\phi)$

so

$$\boxed{\mathcal{L} = \frac{1}{2}m\left[l^2\dot{\phi}^2 + R^2\omega^2 + 2lR\omega\dot{\phi}\sin(\phi-\omega t)\right] - mg(R\sin\omega t - l\cos\phi)}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = ml^2\dot{\phi} + mlR\omega\sin(\phi - \omega t)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = mlR\omega\dot{\phi}\cos(\phi - \omega t) - mgl\sin\phi$$

1 (cont'd). $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\ddot{\phi} + mlR\omega \cos(\phi-\omega t)(\dot{\phi}-\omega)$

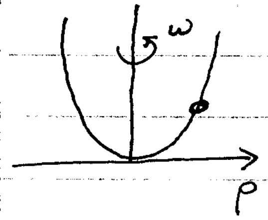
So the Euler-Lagrange eq is

$$ml^2\ddot{\phi} + mlR\omega\cos(\phi-\omega t)(\cancel{\dot{\phi}}-\omega) - mlR\omega\cancel{\dot{\phi}}\cos(\phi-\omega t)$$
$$+ mgl\sin\phi = 0$$

$$\Rightarrow ml^2\ddot{\phi} - mlR\omega^2\cos(\phi-\omega t) + mgl\sin\phi = 0$$

$$\boxed{l\ddot{\phi} = R\omega^2\cos(\phi-\omega t) - g\sin\phi}$$

7.41) 2.



$z = K\rho^2$ It is easiest to use cylindrical coord (ρ, ϕ, z). The potential energy is:

$$U = mgz = mgk\rho^2$$

The kinetic energy is

$$T = \tfrac{1}{2} m \left(v_\rho^2 + v_\phi^2 + v_z^2\right)$$

$v_\phi = \rho\omega$, $v_\rho = \dot{\rho}$, $v_z = \dot{z} = 2K\rho\dot{\rho}$, so

$$\boxed{\mathcal{L} = T - U = \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2\omega^2 + 4K^2\rho^2\dot{\rho}^2\right) - mgk\rho^2}$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = m\rho\omega^2 + 4K^2 m\rho\dot{\rho}^2 - 2mgk\rho$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\dot{\rho} + 4K^2 m\rho^2\dot{\rho}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\ddot{\rho} + 4K^2 m\rho^2\ddot{\rho} + 8K^2 m\rho\dot{\rho}^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = \frac{\partial \mathcal{L}}{\partial \rho} \Rightarrow m\ddot{\rho} + 4K^2 m\rho^2\ddot{\rho} + 8K^2 m\rho\dot{\rho}^2 = m\rho\omega^2 + 4K^2 m\rho\dot{\rho}^2 - 2mgk\rho$$

$$\Rightarrow \boxed{\left(1 + 4K^2 m\rho^2\right)\ddot{\rho} + 4K^2\rho\dot{\rho}^2 = \left(\omega^2 - 2gk\right)\rho}$$

If $\omega^2 \neq 2gk$, the only const ρ soln is $\rho = 0$. This is stable for $\omega^2 < 2gk$ (since $\ddot{\rho} < 0$ for small ρ)

and unstable for $\omega^2 > 2gk$. For $\omega^2 = 2gk$, every point along the wire is an equilibrium position.

(7.49) 3. (a) If $\vec{B} = B\hat{z}$, then $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ has components $A_z = 0$, $A_x = -\frac{1}{2}By$, $A_y = \frac{1}{2}Bx$

So $\vec{\nabla}\times\vec{A}$ has only a z component and its equal to $\partial_x A_y - \partial_y A_x = B$ ✓

If $\vec{A} = \frac{1}{2}B\rho\hat{\phi}$, then $\vec{\nabla}\times\vec{A}$ has only a z component given by $\frac{1}{\rho}\partial_\rho(\rho A_\phi) = \frac{B}{2\rho}\partial_\rho(\rho^2) = B$

(b) From (7.103): $\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - q(V - \dot{\vec{r}}\cdot\vec{A})$

$V = 0$, $\dot{\vec{r}}^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$, $\dot{\vec{r}}\cdot\vec{A} = \rho\dot{\phi}A_\phi$

So $\mathcal{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{q}{2}B\rho^2\dot{\phi}$

$$\frac{\partial\mathcal{L}}{\partial\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}, \quad \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\dot{\rho}$$

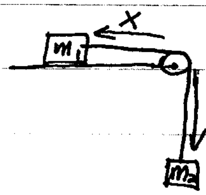
So ρ eq is $\boxed{m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}}$ (*)

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0, \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + \frac{1}{2}qB\rho^2$$

ϕ eq is $\boxed{\frac{d}{dt}\left(m\rho^2\dot{\phi} + \frac{1}{2}qB\rho^2\right) = 0}$

$$\frac{\partial\mathcal{L}}{\partial z} = 0, \quad \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z} \Rightarrow \boxed{m\ddot{z} = 0}$$

(c) From (*), if $\rho = $ const, $\dot{\phi} = -qB/m$. The particle moves in a circle clockwise.

(7.50) 4. 

$f = x + y = \text{const.}$ (constraint)

Unconstrained Lagrangian is

$$\mathcal{L} = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{y}^2 + m_2 g y$$

Modified Euler-Lagrange eqs:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} \Rightarrow m_1 \ddot{x} = \lambda$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} \Rightarrow m_2 \ddot{y} = m_2 g + \lambda$$

So $m_2 \ddot{y} = m_2 g + m_1 \ddot{x}$. But $\ddot{x} = -\ddot{y}$, so

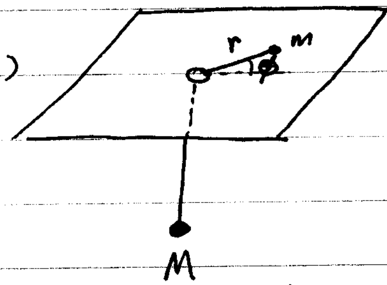
$$\boxed{\ddot{y} = \frac{m_2 g}{m_1 + m_2}}$$

The only force acting on m_1 is the tension, so

$$F_t = m_1 \ddot{x} = \lambda = -\frac{m_1 m_2 g}{m_1 + m_2}$$

since $\ddot{x} = -\ddot{y}$.

You get the same answer using the elementary Newtonian approach since the eqs of motion are identical to above with $\lambda = F_t$.

5. (a) 

Use polar coordinates (r, ϕ) for mass m on table.

Kinetic energy for m is $\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$

Kinetic energy for M is $\frac{1}{2} M \dot{r}^2$

Potential energy for M: $U = Mgr$

So

$$\boxed{\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} M \dot{r}^2 - Mgr}$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\phi}^2 - Mg, \quad \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} + M\dot{r}$$

r eq: $\boxed{(m+M)\ddot{r} = m r \dot{\phi}^2 - Mg}$

ϕ eq: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \boxed{\frac{d}{dt}(m r^2 \dot{\phi}) = 0}$

(b) $\mathcal{L}$ is indep of ϕ, $\Rightarrow L_z$ is conserved

$$L_z = (\vec{r} \times \vec{p})_z = m r^2 \dot{\phi}$$

(c) $r = r_0$ is a solution if $\dot{\phi}^2 = \frac{Mg}{m r_0}$

So

$$\boxed{L_z = m r_0^2 \left(\frac{Mg}{m r_0}\right)^{1/2} = (m M g r_0^3)^{1/2}}$$