Liquids & Solids

1st 2/3 s: of course: Fluids (liquids & gases) rather than solids.

Differ primarily in different response to shearing stresses.

Solid:

- Apply tangential force \( F \).
- Suppose bottom is fixed.
- Surface area \( A \).

How does deformation depend on applied force?

Provided amount of deformation is small, most solid materials obey Hooke's law.

\[
\text{Shear stress} \left[ \frac{F}{A} \right] \times \text{shear strain} = \frac{d^2 x}{dy} = \tan \theta = \Theta
\]

\[
\frac{F}{A} = n \frac{d^2 x}{dy} \approx n \Theta
\]

coefficient of proportionality is shear modulus \( \frac{N}{m^2} \) solid.

N.B. shear stress removed \( \rightarrow \) returns to undeformed shape.

Fluids don't! Remain deformed. Don't have "shape".
Instead, fluids resist rates of deformation.

\[ F = \frac{1}{d} \frac{dv}{dy} = \eta \frac{dv}{dy} \]

Balance between external force driving shear, internal force resisting shear (viscosity).

\[ \frac{F}{A} = \eta \frac{dv}{dy} \]

\[ \text{coefficient of dynamic viscosity} \]

Naturally:
- Fluids can flow.
- But there are idealizations.
- Solids cannot.
  - e.g.
  - (1) Earth’s mantle: solid on short timescales, liquid on long timescales.
  - (2) Glass: same thing (windshields in old houses are thicker at bottom!)

**AIR** can approximate as.

- Ideal gas: (molecully)
  1. Particles occupy negligible volume.
  2. No intermolecular force — perfectly elastic collisions (which are very short time) with one another & walls of container.

Perfect gas equation of state

\[ P = nRT \]

(vs. \( PV = NRT \))
\[ k_B = \text{Boltzmann's constant} = 1.38 \times 10^{-23} \text{ JK}^{-1} \]

Can also rewrite in terms of mass density:
\[ S = n \frac{\mu}{m_n} \]

"mean molecular weight" really a mass not a mole weight
\[ m_n = 1.661 \times 10^{-27} \text{ kg} \]
(atomic mass unit, defined s.t. 12C of pure 12C has mass of exactly 12g)

So
\[ P = \frac{S}{nm_n} k_B T \]

What's \( n \) for air?

Dry Air:
- 78% N\(_2\)
- 21% O\(_2\)
- 1% Ar, CO\(_2\), Ne + trace units of others

If just consider \( N_2 \) & O\(_2\),
\[ n = 0.78(28) + 0.21(32) = 28.6 \]
\[ \approx \frac{14 \times 2}{12 \times 2} \]

If account for other gases, \( n = 28.96 \times (29) \)

Still a good value for humid air. Even when saturated (100% relative humidity — any more liquid droplets only contribute \( \approx 2\% \) of molecules).
\[ \theta C = \frac{5}{9}(\theta F - 32) \]

Pressure at sea level: \( P = 10^5 \text{ Pa} \) \([0.98 - 1.05 \times 10^5 \text{ Pa}] \)

At \( T = 15^\circ C (\approx 59^\circ F) \)

\[
S = \frac{n m u \cdot P}{k B T} = \frac{29 (1.66 \times 10^{-27}) (10^5)}{(1.38 \times 10^{-23}) (15 + 273)}$

\[
= 1.2 \text{ kg m}^{-3}
\]

Rules of small dependence & on temperature:

Rules of thumb:

Air pressure at sea level is \( \sim 10^5 \text{ Pa} \).

Density (at sea level) \( \sim 1 \text{ kg m}^{-3} \)

\[
\frac{S_p}{S} = \left( \frac{2p}{B} \right) \Delta T
\]

\[
= -\frac{8}{T} \Delta T \quad \sim 15
\]

\[
\frac{\Delta T}{300}
\]

Say \( T = 0 - 90 \)

So only \( \approx 5\% \) change

**WATER**

Unlike gases, liquids have const densities over enormous variations in pressure.

For small vol changes, liquids obey a form of Hooke's law:

\[
\Delta P = B \frac{\Delta V}{V_0} \quad \text{bulk modulus}
\]

\[
B \approx 2 \times 10^9 \text{ Pa}
\]

For water, \( B \approx 2 \times 10^9 \text{ Pa} \)

\[
= 1 \text{ N m}^{-2} = \text{Pa}
\]
Can approximate liquid water as **incompressible fluid**.

Rule of thumb: liquid water density \( \rho = 1000 \text{ kg m}^{-3} \approx 1 \text{ g cm}^{-3} \).

*Note: some units.*

- 1 bar \( \approx 10^5 \text{ Pa} \approx 1 \text{ atm} \).
- \( ^\circ \text{C} = \frac{5}{9} (^\circ \text{F} - 32) \).

There are small variations — important for life on earth!

Liquid fresh water max density at 4\( ^\circ \text{C} \).

\( \rightarrow \) this collects at the bottom of frozen ponds/lakes enabling fish to live independent of surface \( \uparrow \uparrow \).

Impurities alter also affect density (saltwater is few \% denser).

**Hydrostatics** [hydro, statics] explains.

Consider fluid at rest in earth \( g \)-field.

\[
\begin{align*}
\text{area } A & \quad \text{d}h \quad P(z + \text{d}z) \\
\text{dz} & \quad \text{Mass} = gA\text{dz} \\
\uparrow P(z) & \quad \text{weight} = gA\text{dz} \quad \uparrow z
\end{align*}
\]

Net pressure force up \( \rightarrow \) \( P(z + \text{d}z)A \uparrow z^{\uparrow} + P(z)A \uparrow z \) — top surface, bottom surface; pressure pushes up, pressure pushes down.

[From symmetry = no net sideways pressure forces.]
\[ \sum F = 0 \text{ for equilibrium} \]

\[-gA/gdz = P(z+dz)/A + P(z)/A = 0 \]

\[ \Rightarrow \lim_{dz \to 0} \frac{P(z+dz) - P(z)}{dz} = -sg \]

\[ \Rightarrow \frac{dp}{dz} = -sg \]

Note -ve sign: pressure decreases as move vertically up.

Consider a lake or ocean:

\[ z \]

\[ z = 0 \]

\[ z = d + \]

\[ \text{Air} \]

\[ P(0) = P_o = \text{atmospheric pressure} \]

\[ \text{Water} \]

\[ \int_{P_o}^{P(D)} dp = - \int_{-d}^{0} -sgdz \]

Water = \text{const} density

\[ g = \text{const} \text{ [ocean depths } \ll \text{radius of earth]} \]

\[ P_o - P(-d) = -sgd \]

\[ \Rightarrow [P(-d) = P_o + gzd] \text{ pressure} \]

Need to support weight of overlying material (water + atm).
How deep to double pressure?

\[ P(-d) = 2 P_0 \Rightarrow P_0 = \frac{\rho gd}{2} \]

\[ d = \frac{P - P_0}{\rho g} = \frac{10^5 \text{ Pa}}{(10^2 \text{ kg m}^{-3})(10 \text{ m s}^{-2})} = 10 \text{ m}. \]

Oceans can be many km deep, so pressures can be 1000s of times atm pressure.

What about going upwards in atmosphere? Density is not const, air is compressible. Need i) eqn of state (assume ideal)

ii) temperature structure (unknown, assume isothermal)

\[ \frac{dp}{dz} = -\rho g \]

\[ P = \frac{\rho k_B T}{n m_u} \]

\Rightarrow \quad s = \frac{n m_u P}{k_B T \rho(2)} \quad \Rightarrow \quad \frac{dp}{dz} = -\frac{n m_u g}{k_B T} P \]

\[ \int_{P_0}^{P} \frac{dp}{p} = -\int_{0}^{z} \frac{n m_u g}{k_B T} \, dz \]

\Rightarrow \quad \ln P(z) - \ln P_0 = -\frac{n m_u g}{k_B T} z.

Scale height
\[ H = \frac{k_B T}{n m u g} \]

\[ P = P_0^{-z^2/4} \]
How does density go?

\[ S = \frac{n m u}{k T} \quad p = \frac{n m u}{k T} \quad P_0 \quad e^{-\frac{2H}{H}} \]

\[ \Rightarrow S = S_0 \quad e^{-\frac{2H}{H}} \]

For \( T = 15^\circ\)C

\[ H = \frac{k T}{n m u g} = \frac{(1.38 \times 10^{-23})(273 + 15)}{29(1.66 \times 10^{-24})(9.8)} = 8.4 \text{ km} \]

Top of Mt. Whitney (highest in lower 48)

14,495 ft = 4.4 km.

Density of air is only \( e^{-4.84} \approx 60\% \) of sea level (that's why it's hard to hike)

In commercial airlines, fly at \( \geq 10 \) km.

Density, pressure only \( e^{-10.84} \approx 30\% \) of sea level value.

What's interpretation of scale height? Have to be pressurized!

Note: \( H = \frac{k T}{n m u g} \)

\[ (n m u) g H = k T \]

potential energy of molecules

Can also estimate it crudely for basic case:

\[ \frac{dP}{dz} = -s g \quad \Rightarrow P = \frac{e k T}{n m u} \]

\[ \Rightarrow \frac{dP}{dz} = -\frac{n m u g}{k T} P \]
But over a scaleheight
\[ \frac{dp}{dz} = -\frac{p}{H} \Rightarrow -\frac{p}{H} = -\frac{n m u g}{k T} \]
\[ \Rightarrow H \sim \frac{kT}{nm u g} \]
Can make quick, order-of-mag. estimates w/out solving eqns.

Also:
\[ p = p_o e^{-z/H} \]
\[ \Rightarrow z = H \ln \left( \frac{p_o}{p(z)} \right) = \frac{kT}{n m u g} \ln \left( \frac{p_o}{p(z)} \right) \]
(hypsoc = kT) hypsometric eqn

Use local air pressure
to measure altitude above sea level.

But: temperature decreases w/ height in the troposphere
(lowest layer of atmosphere), < 12 km.

Average ELR "Environmental lapse rate", is 6.5° per km.

Standard altimeters in aircraft assume this +15°C at
sea level.

Also: airport report local ground conditions so
pilots can reset altimeters.
As GPS becomes more accurate, this ground-based way
becomes obsolete.
Buoyancy

Fluid elements experience buoyancy forces if densities differ. Fluid accelerates vertically.

E.g. hot air balloons
helium

Convection in atmosphere
"oceans (thermohaline system)

diff. in temp., diff. in density, salinity

Consider a fluid element with slightly different density from surrounding.

What is the net buoyancy force?

Net pressure force is $-\int p \, dA$.

Since due to surrounding fluid, independent of density, would be the same for a parcel identical to and consisting of surrounding fluid.

In hydrostatic equilibrium:

$\int \int \int p \, dV = \rho g V \Delta z$

Archimedes Principle: net pressure force is upward, equal to weight of fluid displaced.
Net force
\[ F = \int_P \mathbf{P} \cdot d\mathbf{A} - \mathbf{g} \mathbf{V} z' \]
\[ = -sg \mathbf{V} z' - s \cdot g \mathbf{V} z' \]
\[ = - (s \cdot V) g \left( 1 - \frac{z}{z_i} \right)^2 \]
\[ F = m \mathbf{a} = g \mathbf{V} \mathbf{a} \]

So experiences an acceleration:
\[ \mathbf{a} = -g \mathbf{z} \left( 1 - \frac{z}{z_i} \right) \]

\( s_i < s \quad \Rightarrow \text{up (low density float)} \)
\( s_i > s \quad \Rightarrow \text{down (high "sunk")} \)

Saltwater \( s = 1026 \text{ kg m}^{-3} \)
Fish \( s = 1064 \text{ kg m}^{-3} \).

Fish would tend to sink, \( \mathbf{a} \) with accele\( \text{rataion} \)
\[ \mathbf{a} = g \left( 1 - \frac{1026}{1064} \right) = 0.35 \text{ m s}^{-2} \]

**SCUBA**: They would sink rapidly if did not

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**DIVING**

Adaptation: Swim bladder - sac can fill with gas to achieve neutral buoyancy.
Various types.
Most advanced: disconnected R. fish's exterior, filled chemically through gas gland (dump lactic acid into blood, pH shifts, release oxygen R. hemoglobin).

Slow to fill, can't change depth quickly.
Else, bladder will expand → unstable, runaway (need to swim vigorously down!).
Fishes w/ such bladder found at 7 km, at 700 atm!

Cat: Cartilaginous fishes (sharks) + bony fishes have to swim (no bladders). Flying through ocean.
Need to consider fluid flow.