Laminar Boundary Layers

Make order of magnitude estimate of thickness of boundary layer.

$$\frac{d^2 v}{d t} = \frac{1}{\nu} \frac{d^2 v}{d x^2} \implies \frac{8}{d} \text{diff} \sim \frac{1}{\nu} \frac{v}{\delta^2}$$

$$\implies \frac{d}{t} \text{diff} \sim \frac{8^2}{\nu} = \frac{\delta^2}{v}$$

In this time, the flow has gone a distance

$$x \sim v_0 \cdot t \text{diff} \text{ past the plate edge.}$$

$$\implies \frac{x}{v_0} \sim \frac{\delta^2}{v} \implies \delta \sim \sqrt{\frac{D}{v_0} x} = x \sqrt{\frac{D}{v_0 x}} = x \sqrt{\frac{D}{v_0 x}} = \sqrt{\frac{D}{v_0 x}} = x \sqrt{\frac{D}{v_0 x}} = x (Re_x)^{\frac{1}{2}}$$

where \( Re_x = \frac{v_0 x}{\nu} \) is the Reynolds number associated with the lengthscale \( x \).

Accurate soln to problem is actually.

$$\frac{\delta}{x} \geq \frac{1}{\sqrt[4]{Re_x}} \text{ where } \delta \text{ is defined to be the thickness outside which flow velocity differs by } \pm 1\% \text{ from upstream velocity } v_0.$$ 

Note: \( \delta \propto \frac{1}{v_0} \)

higher speeds correspond to thinner boundary layers, as might be expected.
Example: Flow past hood & windshield of car.
\( v = 70 \text{ mph} = 30 \text{ ms}^{-1} \).
\( x = 1 \text{ m} \).
\( \text{Re}_x = \frac{(v)(30)}{2 \times 10^{-5}} = 1.5 \times 10^4 \).

\[ S = \frac{5x}{\sqrt{\text{Re}_x}} = \frac{5(1)}{\sqrt{1.5 \times 10^6}} = 4 \times 10^{-3} \text{ m} = 4 \text{ mm} \]

Thin!

What is velocity profile inside boundary layer?
For \( y \leq 2.5 \frac{\sqrt{\text{v}_0}}{\text{v}_0} \), accurate soln. gives linear

\[ \text{v}_x = 0.332 \text{ v}_0 y \sqrt{\frac{\text{v}_0}{\text{v}_x}} \]

Hence for a car's windshield,

\[
\begin{array}{c|c|c}
\text{y} & \text{v}_x & \\
1 \text{ mm} & 1.2 \text{ m s}^{-1} & \\
100 \text{ mm} & 1.2 \text{ m s}^{-1} & \\
10 \text{ mm} & 0.12 \text{ m s}^{-1} & \\
1 \text{ mm} & 0.01 \text{ m s}^{-1} & \\
\end{array}
\]

No-slip boundary condition is achieved over very small length scales!

Flow in laminar boundary layer produces tangential drag force on solid surface that's entirely due to viscous friction. Calculate this for flat plate...
Drag force per unit area on each side of plate

\[ \sigma_{xy} = \gamma \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \]

negligible, since \( v_y \ll v_x \) and \( \frac{v_x}{L} \sim \frac{1}{x} \ll \frac{1}{L} \frac{1}{\partial x}. \)

Hence, to order of magnitude

\[ \sigma_{xy} \sim \gamma \frac{v_x v_o}{8} \sim \gamma \frac{v_o \sqrt{v_o}}{5} \frac{v_o}{\partial x} \]

Accurate solution for flow in boundary layer gives:

\[ v_x = 0.332 \frac{v_o y}{\sqrt{v_o / \partial x}} \Rightarrow \frac{\partial v_x}{\partial y} = \frac{v_x}{\sqrt{v_o / \partial x}} \]

\[ F = \int_0^L \int_0^w \sigma_{xy} \, dx \]

\[ = \int_0^L \gamma (0.332) \frac{v_o y}{\sqrt{v_o / \partial x}} \int_0^x x^{-\frac{1}{2}} \, dx \]

\[ = \int_0^L \gamma (0.332) \frac{v_o y}{\sqrt{v_o / \partial x}} (2) \sqrt{L} \]

\[ \Rightarrow F = 0.664 (Lw) \frac{v_o \sqrt{v_o / \partial L}}{\partial L} \]

Can define drag coefficient by same formula that was used for lift coefficient:

\[ F_d = \frac{1}{2} C_d \rho A v_o^2 \]

area on each side of plate = \( Lw \).

\[ C_d = \frac{2 F_d}{\rho (Lw) v_o^2} = 1.328 \gamma v_o \sqrt{v_o / \partial L} = 1.328 \sqrt{\frac{v_o}{v_o / \partial L}}. \]
\[
C_d = \frac{1.328}{\sqrt{\text{Re}_l}}
\]

\[
\text{Re}_l = \frac{V\, L}{\nu}
\]

is Reynolds number over lengthscale L.

Instabilities and Turbulence

We've been assuming that fluid flows are **laminar**; having smooth, well-defined streamlines that only vary over long length and timescales.

Turbulent flows instead have variations on wide range of length & timescales simultaneously, and fluid elements move on chaotic trajectories.

Impossible to solve analytically!

Can still learn lots fr. computer sims + experiment.

Turbulence arises b/cos laminar flow can be **unstable**

& some source of free energy in which perturbations can grow.

**Rayleigh–Taylor**
If \( g_1 > g_2 \), gravitational potential energy released if two fluids exchange positions. Wave on interface would be unstable & grow in amplitude.

Neglect viscosity, assume incompressible, and assume each fluid layer is infinitely thick, dispersion relation is:

\[
\omega^2 = \frac{k g (g_2 - g_1)}{(g_2 + g_1)} + \frac{k^3 \gamma}{(g_2 + g_1)^2} \text{Surface tension}
\]

\[
k = \frac{\sqrt{k_i^2 + k_y^2} - 2\pi}{\lambda}
\]

Note: \( g_1 \ll g_2 \) (air over water) → get old dispersion relation for deep water surface waves

\[
\omega^2 = \frac{k g}{g_2} \left[ \frac{k^3 \gamma}{g_2^2} \right]
\]

Before, we neglected response of air to perturbations in the water → didn’t get full RT dispersion relation. This is gd. approx since \( g_1 \ll g_2 \).

\( g_1 < g_2 \) ⇒ \( \omega^2 > 0 \) for all wavelengths, system is stable.

\( g_1 > g_2 \)

\( \omega^2 < 0 \) if:

\[
\omega^2 = \frac{k g (g_2 - g_1)}{(g_2 + g_1)} + \frac{k^3 \gamma}{(g_2 + g_1)^2} < 0
\]

\[
\Rightarrow \frac{2\pi}{\lambda} = k < \sqrt{\frac{g_1 (g_2 - g_1)}{\gamma}}
\]

\[
\Rightarrow \lambda > 2\pi \sqrt{\frac{\gamma}{g_1 (g_2 - g_1)}}
\]
At long $\lambda$, exponential instability.

Fluids interpenetrate, flow becomes nonlinear & complex.

Short wavelengths stabilized by surface tension $\rightarrow$ while gravitational potential energy would be released by intermixing two fluids, it costs too much surface energy at short $\lambda$.

Example: if cap top end of straw w/ finger, lift column of water $\rightarrow$ straw is narrow enough that surface tension stabilizes Rayleigh–Taylor. How wide can straw be for this to work?

Let straw have radius $R$. Expect water to be stable if minimum unstable wavelength cannot fit inside straw’s diameter, i.e.

$$2R \leq 2\pi \sqrt{\frac{\gamma}{g(\rho_1-\rho_2)}} \Rightarrow R \leq \pi \sqrt{\frac{\gamma}{g(\rho_1-\rho_2)}}$$

Exact soln: $R < 1.84 \sqrt{\frac{\gamma}{g(\rho_1-\rho_2)}}$

$$= 1.84 \sqrt{\frac{0.07}{9.8(\text{g/cm}^3)}} = 0.5\text{ cm}$$

So if diameter $< 1\text{ cm}$, can lift water.
(2) Kelvin-Helmholtz

Consider uniform density fluid with discontinuity in velocity.

Free energy in form of relative velocity between fluid layers. Wave on interface which acts to mix momentum between layers can be unstable.

Dispersion relation for this problem turns out to be:

\[ \omega = k_y \left( \frac{v_1 + v_2}{2} \right) + i k_y \left( \frac{v_1 - v_2}{2} \right) \]

- Real part = waves with group velocity equal to average fluid velocity across interface.
- Imaginary part = instability w/ exponential growth

\[ \exp \left[ \frac{k_y \Delta v}{2} t \right] \]

Hence these are unstable, propagating waves.

Again the fluids interpenetrate and flow becomes complex.

Faster moving fluid slows down, slower moving fluid speeds up.

Unstable waves can be seen in “billow cloud” formations produced by wind shear in atmosphere.

K-H is simplest example of shear flow instability.

Since viscosity is stabilizing, Reynolds number is important in determining if flow is unstable to development of turbulence.
Consider pipe flow.

low Reynolds # → get classic H-P flow w/ parabolic shear profile.

High Reynolds # → unstable, transition to turbulence.

Low Re → dye streaking

High Re

Flow becomes turbulent for Re ~ 2000 - 10^5, exact value depends on entry conditions into the pipe.

When flow becomes turbulent, causes large increase in viscosity, as "eddies" in turbulence efficiently mix momentum.

Eddies on broad range of scale, small eddies are carried around by bigger eddies.
For homogeneous, isotropic, 3D incompressible turbulence, can calculate shear eddy velocity $v_x$ on length scale $\lambda$ using energy conservation. Large $\rightarrow$ small scales, then dissipated as heat.

Energy at each scale (per unit mass) $\frac{1}{2} v_x^2$.

Transferred to smaller scales on eddy turnover time $t = \frac{1}{v_x}$

\[
E = \frac{1}{2} v_x^2 = \frac{v_x^3}{2 \lambda} = \text{constant as energy is conserved, turbulence has constant statistical prop. over time.}
\]

\[
\implies v_x \propto \lambda^{-\frac{1}{3}}, \quad \text{smaller eddies rotate at smaller linear speeds.}
\]

Reynolds number at scale $\lambda$

\[
Re_\lambda = \frac{\lambda v_x}{\nu} \propto \lambda^{\frac{4}{3}} \quad \text{decreases down cascade until}
\]

\[
\text{Re} \rightarrow 1, \text{viscosity clamps out eddy motion.}
\]

If $Re_\lambda$ is Reynolds # on largest macroscopic scale $L$, then viscous scale given by

\[
\lambda_{visc} = L \left(Re_\lambda\right)^{-\frac{3}{4}}
\]
Cascade mixes gradients in quantities from large to small scales.

E.g. mixing of cream in coffee cup. Mixing by viscosity is slow, but turbulent cascade caused by stirring mixes stuff rapidly.

Similarly, turbulence causes mixing of momentum—greatly enhances viscosity of fluid. Think of this as eddy viscosity caused by turbulence:

\[ d_{\text{turb}} \sim \frac{v}{L} \quad \text{(roughly)} \]
\[ \frac{d_{\text{turb}}}{d} \sim \frac{v}{L} = \frac{\text{Re}_L}{d} \quad \text{which can be very large.} \]

Similarly, we can talk about turbulent shear stress:

\[ \sigma \sim \frac{\rho d_{\text{turb}}}{\mu} \left( \frac{v}{L} \right) \sim \frac{\rho}{\mu} \left( \frac{v}{L} \right) \left( \frac{v}{E} \right) \]

Consider pipe flow. \[ \text{Re} = \frac{vR}{\nu} \ll 1 \quad \text{(low Reynolds #)} \]
then have Hagen–Poiseuille sol'n:

\[ J = \frac{\pi R^4}{8} \left( \frac{\Delta P}{L} \right) \]

\[ \text{Re} = \frac{vR}{\nu} \Rightarrow v = \frac{\text{Re}_L}{d} \]
\[ J = \pi R^2 v = \pi R^2 \left( \frac{v}{R} \text{Re} \right) = \pi R^2 \cdot \text{Re} \]
\[ \Rightarrow \text{Re} \pi R^3 = \frac{\pi R^4}{8y} \left( \frac{\Delta P}{L} \right) \]

\[ \Rightarrow \text{Re} = \frac{R^3}{8y} \left( \frac{\Delta P}{L} \right) = \frac{R^3 \rho}{8y^2} \left( \frac{\Delta P}{L} \right) \]

Dimensionless measure of flow velocity in pipe

Dimensionless measure of pressure gradient in the pipe.

For pipe of fixed radius, pressure gradient required to produce a given average flow velocity is directly proportional to that velocity:

\[ \frac{\Delta P}{L} \propto \text{Re} \propto v \text{ for fixed } R, y. \]

For high Reynolds numbers, \( y \rightarrow y_{	ext{turb}} \propto \sqrt{\text{Re}} \).

Hence

\[ J = \pi R^3 \text{Re} = \frac{\pi R^4}{8y} \left( \frac{\Delta P}{L} \right) \]

\[ \Rightarrow (\text{Re})^2 \propto \frac{R^3 \rho}{8y^2} \left( \frac{\Delta P}{L} \right) \]

\[ \Rightarrow \left( \frac{\Delta P}{L} \right) \propto (\text{Re})^2 \propto v^2 \text{ for fixed } R, y. \]
Another viewpoint:
\[ \delta \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla \rho + \frac{1}{\rho} \nabla^2 \mathbf{v} \]
\[ = \nabla \text{turbulent drag} + \nabla \text{viscous drag.} \]

For low Re, \( \frac{\Delta P}{L} + \frac{1}{\rho} \frac{v}{R^2} \)
\[ \times \frac{R^3 h}{Y^2} , \quad \frac{R^3 h}{Y^2} \left( \frac{\Delta P}{L} \right) = \frac{R^3 h}{Y^2} \frac{v}{R^2} \]
\[ \sim \frac{Re v}{Y} = Re \]

For high Re
\[ \frac{Re}{R} \sim \frac{\Delta P}{L} \]
\[ \Rightarrow \frac{R^3 h}{Y^2} \frac{\Delta P}{L} \sim \theta \frac{R^3 h}{Y^2} \frac{Re v^2}{R} \]
\[ \sim \frac{R^2 v^2 \frac{v^2}{Y^2}}{Y^2} = (Re)^2 \]

Flow Past Cylinders & Spheres as a function of Reynolds Number