Physics 123B: Homework 5

due February 27, 4pm to Amanda Jones in Kohn Hall 1220 or by email to Prof. Balents

1. **Flux quantum:** Suppose experimentalists measure the magnetic flux of vortices in a new superconducting material. They find this flux is equal to $\pm h/4e$. What does this imply about the nature of superconductivity in this material?

   This means that instead of Cooper pairs, the superconductor consists of a condensate of "quartets" of *four* electrons.

Homework 5 - Solutions

Problem 2 - Fraunhofer Pattern

To start, we note that we write $\gamma(x)$ as

$$\gamma(x) = \nabla\theta - \frac{2e}{\hbar}\int_1^2 \vec{A}\cdot d\vec{\ell} \tag{1}$$

In our case $\nabla\theta = \text{const}$ and $\vec{A} = -Bx\,\hat{z}$. Thus

$$\begin{aligned} \gamma(x) = \nabla\theta - \frac{2e}{\hbar}\int_0^d (-Bx)dz &= \nabla\theta - \frac{2Bed}{\hbar}x && (2)\\ &= \nabla\theta - \frac{2\pi Bd}{\varphi_0}x && (3) \end{aligned}$$

Where $\varphi_0 = \frac{h}{2e}$

The total flux through the josephson junction is

$$\begin{aligned} \Phi &= \oint \vec{A}\cdot d\vec{\ell} && (4)\\ &= \int_0^d A_z(x = L_x)dz \; + \int_d^0 A_z(x=0)dz && (5)\\ &= -BL_x(d) \; + 0 && (6)\\ \Rightarrow \quad \Phi &= -BL_x d && (7) \end{aligned}$$

Therefore

$$\gamma(x) = \Delta\theta + \frac{2\pi\,\Phi}{L_x\varphi_0}x \tag{8}$$

Now, we have that the total current is given by

$$\begin{aligned} I &= \int_0^{L_x} j_c \sin\big(\gamma(x)\big) && (9)\\ &= \int_0^{L_x} j_c \sin\left(\Delta\theta \; + \; \frac{2\pi\Phi}{L_x\varphi}\,x\right)dx && (10)\\ &= -j_c\frac{L_x\varphi_0}{2\pi\,\Phi}\left[\cos\left(\Delta\theta \; + \; \frac{2\pi\Phi}{L_x\varphi}\,x\right)\right]_0^{L_x} && (11)\\ &= j_c\frac{L_x}{2\pi}\frac{\varphi_0}{\Phi}\left(\cos\left(\Delta\theta\right) - \cos\left(\Delta\theta + \frac{2\pi\Phi}{\varphi_0}\right)\right) && (12) \end{aligned}$$

Now we write:

$$\begin{aligned}
\cos(\Delta\theta) - \cos(\Delta\theta + a) &= \cos(\Delta\theta) - \cos(\Delta\theta)\cos(a) + \sin(\Delta\theta)\sin(a) && (13)\\
&= \cos(\Delta\theta)[1 - \cos(a)] + \sin(\Delta\theta)\sin(a) && (14)\\
&= \cos(\Delta\theta)[2\sin^2(a/2)] + 2\sin(\Delta\theta)\sin(a/2)\cos(a/2) && (15)\\
&= 2\sin(a/2)[\cos(\Delta\theta)\sin(a/2) + \sin(\Delta\theta)\cos(a/2)] && (16)\\
&= 2\sin(a/2)[\sin(\Delta\theta + a/2] && (17)
\end{aligned}$$

Therefore, using this identity in our equation for current gives

$$I = j_c \frac{L_x}{\pi} \frac{\varphi_0}{\Phi} \sin\left(\frac{\pi\Phi}{\varphi_0}\right) \sin\left(\Delta\theta + \frac{\pi\Phi}{\varphi_0}\right) \tag{18}$$

Therefore, clearly the max current is:

$$j_c \frac{L_x}{\pi} \frac{\varphi_0}{\Phi} \sin\left(\frac{\pi\Phi}{\varphi_0}\right) \tag{19}$$

Problem 3 - Hund's Rule

For an ion which has 4 electrons in its d shell, Hund's first rule tell's us that we want to maximize the total spin. Sine the d shell has orbital quantum number $\ell \in \{-2, 2\}$, there are five possible angular momentum state we can put the electron in. Then we can safely put all 4 electrons in the spin up $s = +1/2$ states so that total spin is $S = 2$.

Hund's second rule states that we want to maximize total angular momentum L_z. Each electron can have $m_\ell \in \{-2, \ldots, 2\}$. Pauli's exclusion principle states states that we can only have one spin up electron for each value of m_ℓ so that the maximum total L^z that we can assign to the group of 4 electrons $m_\ell = 2, 1, 0, -1$. Then the overall angular momentum is $L = 2 + 1 + 0 + (-1) = 2$. Finally, the d shell can hold 10 electrons, so it is less than half filled, so that Hund's third rule states that $J = |L - S| = |2 - 2| = 0$.

Therefore $S = 2$, $L = 2$ and $J = 0$. Or in spectroscopic notation

$$^{2S+1}L_J \;=\; {}^5P_0$$

(Where P represents the $L = 2$ state in this notation).

Now, if we instead consider an ion with 2 f electrons in its outer shell, in this case each angular quantum number can take on any value from $\ell = \{-3, \ldots, 3\}$, so that the shell can hold 14 possible electrons. Then, Hund's first rule implies that all electrons are spin up $s = +1/2$ so that $S = 3/2$.

Hund's second rule says we want to assign the angular numbers $m_\ell = 3, 2$ and 1 to our three electrons so that the total angular momentum is $L = 3 + 2 + 1 = 6$.

Once again, our outer shell is less than half full so that Hund's third rule states that $J = |L - S| = |6 - \frac{3}{2}| = \frac{9}{2}$

Therefore our ground state would be the state with $S = \frac{3}{2}$, $L = 6$ and $J = \frac{9}{2}$

In spectroscopic notation (noting that $L = 6$ is represented by a capital I), this groundstate is denoted by

$$^{2S+1}L_J \;=\; {}^4I_{9/2}$$