Physics 123B: Final
Due March 16, 2015, 9:30AM

1. Imagine that electrons have spin $S = 1$ instead of $S = 1/2$, so that there are 3 possible spin states for a single electron. *Neglecting* the Zeeman interaction, what would be the values of the Hall conductivity, in units of e^2/h, in the quantum Hall regime for a two-dimensional electron gas? For graphene?

   Without Zeeman interactions, all the spin states are degenerate, and so for a 2DEG the Landau levels are three-fold degenerate. Hence the Hall conductivity is a multiple of 3 e^2/h, starting from zero corresponding to no filled Landau levels,i.e. $\sigma_{xy} = 3ne^2/h$. In graphene, there is an additional 2-fold valley degeneracy, so that the LLs are 6-fold degenerate. Taking into account that the neutral system has $\sigma_{xy} = 0$ and that half of this LL gives rise to upward bending edge states (and half downward bending edge states), we get that $\sigma_{xy} = \pm 3e^2/h, \pm 9e^2/h, \cdots = \pm(6n+3)e^2/h$.

2. We can expect that superfluidity breaks down if the phase gradient is too large. For Helium, a reasonable guess would be that this occurs when the phase varies by 2π over $1\mathring{A}$, comparable to the inter-atomic distance. What is the "critical velocity"?

   So the superfluid velocity is $v = \hbar/m\nabla\theta$, hence we estimate $v_c \approx 2\pi\hbar/(ma_0) = h/(ma_0)$, where $m = 6.65 \times 10^{-27}kg$ is the mass of the helium atom, $h = 6.62 \times 10^{-34}Js$, and $a_0 = 1\mathring{A} = 10^{-10}m$. So $v_c \approx 1000m/s$.

3. In class we explained why a flow of superfluid helium in an annular container can be almost infinitely long-lived ("persistent"). What about the flow of superfluid helium in a bucket without a hole in the middle? Is it also long-lived? How does it decay?

   We do not expect it to be long-lived, because it can decay by simply moving the existing vortices in the rotating liquid to the edge of the bucket and allowing them to "escape" out the side of the container. There is no energy barrier for them to do this.

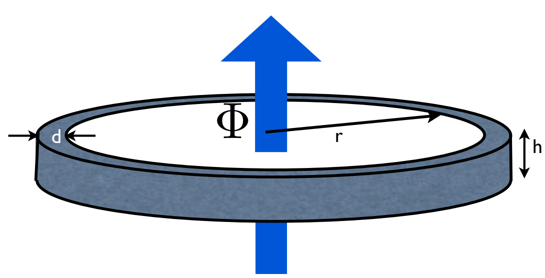

Figure 1: Superconducting ring from problem 4.

4. Consider a thin superconducting ring (Fig. 1), for which the inner and outer radius differ by much less than the penetration depth λ. Under this condition, the magnetic field is constant in the material. Suppose a solenoid is introduced inside the ring. Using London

theory, find and plot the minimum free energy of the superconductor, $F_{\min}(\Phi)$, as a function of the flux Φ it produces. You should assume the field of the solenoid is confined well inside the ring, so that no field penetrates the superconductor, and also that $n_s = \overline{n_s}$. Take the inner radius of the ring to be r, the outer to be $r+d$, and the height of the ring to be h. Hint: your answer $F(\Phi)$ should have slope discontinuities.

Following the problem definition, we take $B = 0$ and $n_s = \overline{n_s}$ inside the superconductor. The London free energy then depends upon the phase θ and in principle the vector potential $\vec{A}$. Since the field itself is zero, $\nabla \times \vec{A} = 0$, but since there is non-zero flux inside, we have $\oint \vec{A} \cdot d\vec{\ell} = \Phi$, when integrating a loop enclosing the hole of the ring. This is satisfied by a vector potential which is oriented in the azimuthal direction. Adopting cylindrical coordinates r, ϕ, z, we have $A_\phi(r) = \Phi/(2\pi r)$. We take r as approximately constant over the ring, since $d \ll r$. Under these assumptions, we expect that the superconducting phase θ is a function of the azimuthal angle ϕ only, and so $\vec{\nabla}\theta = 1/r\partial_\phi\theta\hat{\phi}$. Then the London free energy is

$$\begin{aligned} F &= \int_{ring} \left[\frac{n_s}{8m} |\hbar\nabla\theta + 2e\vec{A}|^2 + a(n_s - \overline{n}_s)^2 + \frac{B^2}{2\mu_0} \right] \\ &= \int_0^h dz \int_r^{r+d} dr r \int_0^{2\pi} d\phi \frac{n_s}{8mr^2} (\hbar\theta'(\phi) + e\Phi/\pi)^2 \\ &\approx \frac{h 2\pi\hbar^2 n_s}{8m} \frac{d}{r} \left(\theta'(\phi) + e\Phi/(\pi\hbar)\right)^2 . \end{aligned} \tag{1}$$

Now we should minimize this over $\theta'(\phi)$. If possible, we would like to choose $\theta'(\phi) = -e\Phi/(\pi\hbar)$, but this means $\theta(\phi) = -e\Phi/(\pi\hbar)\phi + \theta(0)$, which in general is not allowed because the condensate wavefunction must be single valued, so $\theta(\phi + 2\pi)$ must differ from $\theta(\phi)$ by a multiple of 2π. So the best we can do in general is to take $\theta(\phi) = -n\phi + \theta(0)$, with some integer n. Then the energy becomes

$$F = \frac{h 2\pi\hbar^2 n_s}{8m} \frac{d}{r} \left(\frac{\Phi}{\phi_0} - n \right)^2 , \tag{2}$$

where $\phi_0 = \pi\hbar/e$ is the superconducting flux quantum. We are free to choose n to minimize

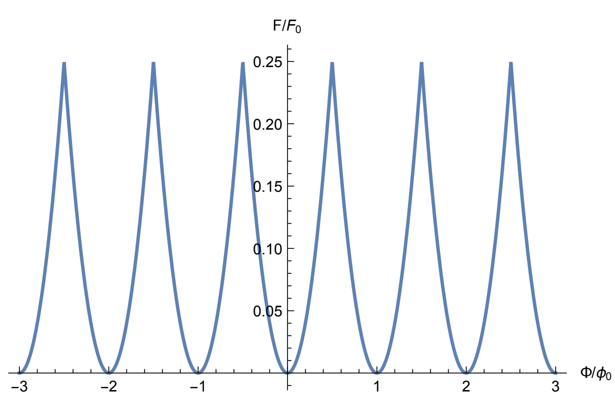

Figure 2: Free energy of the ring in Problem 4.

F, so the real answer is

$$F = F_0 \min_n \left(\frac{\Phi}{\phi_0} - n \right)^2 . \tag{3}$$

where $F_0 = \frac{hd\pi\hbar^2 n_s}{4mr}$. This is clearly a periodic function of Φ with period ϕ_0. It is equal to zero when Φ/ϕ_0 is an integer (and we should choose n equal to that integer), and is maximum when Φ/ϕ_0 is an half-integer, at which point the best n is degenerate: it can be either the integer above or below this half-integer. So the result looks like a set of parabolas, attached at the top.

The rest is a note to people who started by setting $\theta = 0$ by a "gauge choice". This is not allowed in the present problem. The reason is that to do this, one really must make a gauge transformation $\vec{A} \to \vec{A} - \vec{\nabla}\chi$, where in the interior of the superconductor, $\vec{\nabla}\chi = \frac{\hbar}{2e}\vec{\nabla}\theta$. However, *if* θ winds around the ring, then so must χ, and there is no function χ which is non-singular everywhere in the space inside the ring. Equivalently, $\vec{\nabla}\chi$ would have to diverge somewhere inside the ring. You can also see that this is not allowed because making this "gauge" transformation will change a physical quantity, the flux $\Phi = \oint \vec{A} \cdot d\vec{\ell}$ through the loop, so it cannot be "pure gauge".

The problem is tricky because you got used to the idea of flux quantization in a superconductor. But really this only holds for magnetic fields which are entirely surrounded by a *thick* superconductor whose width is large compared to λ. Otherwise it is not possible for enough screening currents to form to reach the quantized flux. By assuming $d \ll \lambda$ in this problem, there is no quantization. Rather there is a small tendency of the free energy to prefer values of the flux which correspond to the quantized values that would be reached for a thick ring.

5. As the magnetic field is increased within the vortex lattice phase of a type II superconductor, does the spacing between vortices increase or decrease, and why?

   The spacing decreases as field is increased. The flux per vortex is fixed, so more vortices must enter a given area to accomodate the increasing flux.

6. Find the S, L, and J quantum numbers for Os^{3+}, Os^{4+}, and Os^{5+} ions in free space, assuming that the first electrons Os loses when it is ionized are its two 6s ones.

   The electronic configuration of neutral Os is [Xe]$4f^{14}5d^6 6s^2$. Losing two electrons, one obtains that Os^{2+} has 6 5d electrons. So Os^{3+}, Os^{4+} and Os^{5+} have 5,4, and 3 outer d electrons, respectively.

   In the case of Os^{3+} with 5 d electrons, the d shell is half filled. This leads via Hund's rule I to total spin S=5/2. Then L=0 since all states have one electron, and hence J=S=5/2.

   In the case of Os^{4+} with 4 d electrons, we obtain S=2. We fill 4 orbitals, leading to L=2. Since the shell is less than half-filled, we should minimize J and so J=0.

   In the case of Os^{5+} with 3 d electrons, we obtain S=3/2. We fill 3 orbitals, leading to L=2+1+0=3. Since the shell is less than half-filled, we should minimize J and so J=3-3/2=3/2.

7. Consider the $S = 1/2$ *quantum* Ising model in a transverse field, defined by

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h_\perp \sum_i S_i^x. \tag{4}$$

   Assume that the lattice consists of all identical sites, with z nearest-neighbors per site. Note that, unlike in the Heisenberg model, only the z components of spins couple between nearest-neighbor sites. Here $\vec{S}_i$ is the usual spin-1/2 operator with $\vec{S}_i^2 = S(S+1) = 3/4$ with $S = 1/2$, the eigenvalues of $S_i^z = \pm 1/2$, etc. (we set $\hbar = 1$).

(a) Apply the mean-field approximation to decouple the first term to obtain a set of spins in an effective field $\vec{h}_{\text{eff}} = (h^x_{\text{eff}}, 0, h^z_{\text{eff}})$. Find h^z_{eff} in terms of the Ising magnetization $m \equiv \langle S^z_i \rangle$.

To carry out the MF approximation, we decouple the J term just as we did in class. So we replace

$$\begin{aligned} H &\to -J\sum_{\langle ij\rangle}\left[\langle S^z_i\rangle S^z_j + S^z_i\langle S^z_j\rangle - \langle S^z_i\rangle\langle S^z_j\rangle\right] - h_\perp\sum_i S^x_i. \\ &= \sum_i\left[-JzmS^z_i - h_\perp S^x_i\right] \\ &\equiv -\sum_i \vec{h}_{\text{eff}}\cdot\vec{S}_i, \end{aligned} \tag{5}$$

where we dropped a constant in the third line. Comparing, we have

$$h^z_{\text{eff}} = Jzm. \tag{6}$$

(b) At *zero temperature*, you can assume each spin is aligned fully by its effective field. Using this assumption, find the self-consistent equation for m.

The ground state of the $-\vec{h}_{\text{eff}}\cdot\vec{S}_i$ term is just the state where each spin is polarized into its maximal "length" eigenstate of $1/2$ along the $\hat{h}_{\text{eff}}$ axis. In other words, we choose a new quantization axis for spin along this effective field. Then then expectation of the components normal to this axis is zero, and along the field it is $1/2$. Thus

$$\langle\vec{S}_i\rangle = \frac{1}{2}\hat{h}_{\text{eff}} = \frac{\vec{h}_{\text{eff}}}{2|h_{\text{eff}}|}. \tag{7}$$

Writing out the z component, we have

$$m = \langle S^z_i\rangle = \frac{h^z_{\text{eff}}}{2\sqrt{(h^x_{\text{eff}})^2 + (h^z_{\text{eff}})^2}} = \frac{Jzm}{2\sqrt{h^2_\perp + (Jzm)^2}}. \tag{8}$$

The equality between the left hand side and the right hand side of this equation is the self-consistent condition.

(c) Solve the above equation to find $m(h_\perp)$. What is the critical field $h^c_\perp$ above which $m = 0$? This is a *quantum critical point*.

We can just solve this by dividing both sides by m, which is ok if $m \neq 0$, and then obtain

$$\sqrt{h^2_\perp + (Jzm)^2} = \frac{Jz}{2}. \tag{9}$$

Squaring both sides and solving gives

$$m = \pm\frac{1}{2}\sqrt{1 - \left(\frac{2h_\perp}{Jz}\right)^2}. \tag{10}$$

This solution works for field $h_\perp$ small enough that the argument of the square root is positive. Beyond that, we must have $m = 0$. So we have $h^x_\perp = Jz/2$.

8. Consider the "frustrated ferromagnetic chain", described by the Hamiltonian

$$H = \sum_{i=-\infty}^{\infty} \left[-J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \right]. \tag{11}$$

Here $\vec{S}_i$ are spin-1/2 spins, and $J_1, J_2 > 0$. For small enough J_2, the ground state is ferromagnetic.

(a) Assuming a ferromagnetic state, calculate the spin wave spectrum of the model.

Following the treatment in class, we can consider the set of spin-flip states $|i\rangle$, where the spin on site i is down while all others are up. As before, let us break up $H = H_z + H_\pm$. Acting with H_z, we have

$$H_z|i\rangle = \left[E_0 + 2 \times J_1 \times \frac{1}{2} - 2 \times J_2 \times \frac{1}{2} \right] |i\rangle, \tag{12}$$

where one factor of 2 comes from site i interacting with sites to its left and right, and the factor of 1/2 comes from the fact that the $S^z S^z$ product on each bond is changed from +1/4 to -1/4.

Now we consider the raising and lowering terms:

$$H_\pm|i\rangle = -\frac{J_1}{2} \left(|i+1\rangle + |i-1\rangle \right) + \frac{J_2}{2} \left(|i+2\rangle + |i-2\rangle \right). \tag{13}$$

Note the sign change and that the spin flip "walks" one step to the right or left with J_1, and two steps to right or left with J_2. Putting it together, we have that

$$H|i\rangle = (E_0 + J_1 - J_2)|i\rangle - \frac{J_1}{2} \left(|i+1\rangle + |i-1\rangle \right) + \frac{J_2}{2} \left(|i+2\rangle + |i-2\rangle \right). \tag{14}$$

Now we seek an eigenstate by writing $|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{ikna} |n\rangle$, where a is a lattice spacing (you could perfectly well take $a = 1$ to make things simpler) . Plugging this in and cancelling exponentials, one gets that

$$H|k\rangle = (E_0 + J_1 - J_2 - J_1 \cos ka + J_2 \cos 2ka)|k\rangle \equiv (E_0 + \epsilon(k))|k\rangle, \tag{15}$$

where the excitation energy or dispersion relation is

$$\epsilon(k) = J_1 - J_2 - J_1 \cos ka + J_2 \cos 2ka. \tag{16}$$

Note that an important check is that, by Goldstone's theorem, this must go to zero as $k \to 0$. Indeed it does!

(b) In this way, determine the maximum value of J_2 for which the ferromagnetic state can be the ground state. At this critical value, $J_2 = J_2^{\max}$, how does the energy depend upon k for small k?

So we need to ask what is the condition that $\epsilon(k) \geq 0$? We can check this by rewriting $\cos 2ka = 2\cos^2 ka - 1$ to give

$$\epsilon(k) = J_1 - 2J_2 - J_1 \cos ka + 2J_2 \cos^2 ka. \tag{17}$$

We can seek the minimum of this by differentiating with respect to k and setting the answer to zero. This gives either $k = 0$ *or* $\cos ka = J_1/(4J_2)$. The latter can be satisfied only when $J_2 \geq J_1/4$. So when $J_2 < J_1/4$, $k = 0$ is definitely the minimum energy, and we saw already that $\epsilon(0) = 0$. So the ferromagnetic state is stable when $J_2 < J_1/4$. What happens when $J_2 > J_1/4$? Well, then we can find the energy when $\cos ka = J_1/(4J_2)$, which is $\epsilon = J_1 - 2J_2 - \frac{J_1^2}{8J_2}$. This is negative when $J_2 > J_1/4$, so we find, finally that

$$\min_k \epsilon(k) = \begin{cases} 0 & J_2 < J_1/4 \\ J_1 - 2J_2 - \frac{J_1^2}{8J_2} < 0 & J_2 > J_1/4. \end{cases} \tag{18}$$

Hence we find that

$$J_2^{\max} = \frac{J_1}{4}. \tag{19}$$

Now we can Taylor expand around $k = 0$. This gives

$$\epsilon(k) \approx_{|k|\ll 1} (\frac{J_1}{2} - 2J_2)(ka)^2 + (-\frac{J_1}{24} + \frac{2J_2}{3})(ka)^4. \tag{20}$$

The first term becomes negative when $J_2 > J_1/4$, as expected. Moreover, setting $J_2 = J_1/4$ in Eq. (20), we see that at this point,

$$\epsilon(J_2 = J_1/4) \approx \frac{J_1}{8}(ka)^4, \tag{21}$$

varying as the fourth power rather than the square of k.

(c) Now add a magnetic field, $H \to H - h\sum_i S_i^z$. A sufficiently large $h > h_c > 0$ stabilizes the ferromagnetic state even when $J_2 > J_2^{\max}$. Find h_c from the condition that the spin wave energies are positive.

We need to a priori repeat the analysis in part (a). The additional term just adds to H_z. Relative to the ferromagnetic (all up) state, this term just adds the energy h to state $|i\rangle$, since a single spin is flipped from up to down. So we have the energy

$$\epsilon(k, h) = h + J_1 - J_2 - J_1 \cos ka + J_2 \cos 2ka. \tag{22}$$

When $J_2 < J_1/4$, we saw that the minimum energy was zero without the field, so the minimum energy with the field is just h, which is always positive, so the ferromagnetic state is always stable, i.e. $h_c = 0$ for $J_2 < J_1/4$.

Now if $J_2 > J_1/4$, we the minimum of $\epsilon(k, 0)$ is negative, and given in Eq. (18). So the field must balance this. We $\epsilon(k, h) > 0$ or $h > -\epsilon(k, 0)$ for all k, hence

$$h > h_c = -\left(J_1 - 2J_2 - \frac{J_1^2}{8J_2}\right) = 2J_2 - J_1 + \frac{J_1^2}{8J_2}. \tag{23}$$

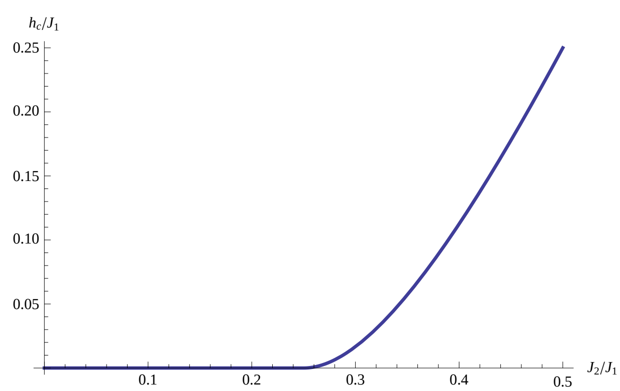

Figure 3: Critical field to stabilize the ferromagnetic state in problem 8c