Physics 123B: Final
Due March 16, 2015, 9:30AM

1. Imagine that electrons have spin $S = 1$ instead of $S = 1/2$, so that there are 3 possible spin states for a single electron. *Neglecting* the Zeeman interaction, what would be the values of the Hall conductivity, in units of e^2/h, in the quantum Hall regime for a two-dimensional electron gas? For graphene?

2. We can expect that superfluidity breaks down if the phase gradient is too large. For Helium, a reasonable guess would be that this occurs when the phase varies by 2π over $1\mathring{A}$, comparable to the inter-atomic distance. What is the "critical velocity"?

3. In class we explained why a flow of superfluid helium in an annular container can be almost infinitely long-lived ("persistent"). What about the flow of superfluid helium in a bucket without a hole in the middle? Is it also long-lived? How does it decay?

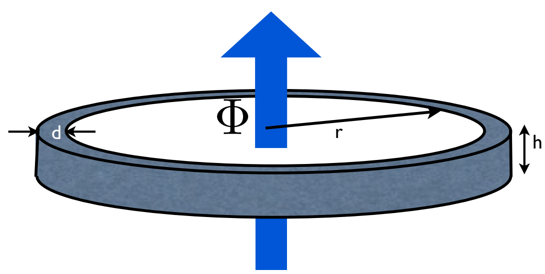

Figure 1: Superconducting ring from problem 4.

4. Consider a thin superconducting ring (Fig. 1), for which the inner and outer radius differ by much less than the penetration depth λ. Under this condition, the magnetic field is constant in the material. Suppose a solenoid is introduced inside the ring. Using London theory, find and plot the minimum free energy of the superconductor, $F_{\min}(\Phi)$, as a function of the flux Φ it produces. You should assume the field of the solenoid is confined well inside the ring, so that no field penetrates the superconductor, and also that $n_s = \overline{n_s}$. Take the inner radius of the ring to be r, the outer to be $r + d$, and the height of the ring to be h. Hint: your answer $F(\Phi)$ should have slope discontinuities.

5. As the magnetic field is increased within the vortex lattice phase of a type II superconductor, does the spacing between vortices increase or decrease, and why?

6. Find the S, L, and J quantum numbers for Os^{3+}, Os^{4+}, and Os^{5+} ions in free space, assuming that the first electrons Os loses when it is ionized are its two 6s ones.

7. Consider the $S = 1/2$ *quantum* Ising model in a transverse field, defined by

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h_\perp \sum_i S_i^x. \tag{1}$$

Assume that the lattice consists of all identical sites, with z nearest-neighbors per site. Note that, unlike in the Heisenberg model, only the z components of spins couple between nearest-neighbor sites. Here $\vec{S}_i$ is the usual spin-1/2 operator with $\vec{S}_i^2 = S(S+1) = 3/4$ with $S = 1/2$, the eigenvalues of $S_i^z = \pm 1/2$, etc. (we set $\hbar = 1$).

   (a) Apply the mean-field approximation to decouple the first term to obtain a set of spins in an effective field $\vec{h}_{\text{eff}} = (h_{\text{eff}}^x, 0, h_{\text{eff}}^z)$. Find h_{eff}^z in terms of the Ising magnetization $m \equiv \langle S_i^z \rangle$.

   (b) At *zero temperature*, you can assume each spin is aligned fully by its effective field. Using this assumption, find the self-consistent equation for m.

   (c) Solve the above equation to find $m(h_\perp)$. What is the critical field $h_\perp^c$ above which $m = 0$? This is a *quantum critical point.*

8. Consider the "frustrated ferromagnetic chain", described by the Hamiltonian

$$H = \sum_{i=-\infty}^{\infty} \left[-J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \right]. \tag{2}$$

Here $\vec{S}_i$ are spin-1/2 spins, and $J_1, J_2 > 0$. For small enough J_2, the ground state is ferromagnetic.

   (a) Assuming a ferromagnetic state, calculate the spin wave spectrum of the model.

   (b) In this way, determine the maximum value of J_2 for which the ferromagnetic state can be the ground state. At this critical value, $J_2 = J_2^{\max}$, how does the energy depend upon k for small k?

   (c) Now add a magnetic field, $H \to H - h \sum_i S_i^z$. A sufficiently large $h > h_c > 0$ stabilizes the ferromagnetic state even when $J_2 > J_2^{\max}$. Find h_c from the condition that the spin wave energies are positive.