Bose-Einstein Condensation

- Minimum density

$$\rho \leq \frac{1}{2\pi^3\hbar^3}\left(\frac{m}{\beta}\right)^{3/2}\sqrt{\frac{\pi}{2}}\zeta(3/2)$$

- At high temperature, i.e. small β, this is always satisfied. But below some T it is not. Defines

$$k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3} \approx 3.31\frac{\hbar^2\rho^{2/3}}{m}$$

- What happens for $T<T_c$?

BEC

- Below T_c, the chemical potential gets "almost" to zero. Then n(k=0) becomes *macroscopic*
- So for $T<T_c$, a macroscopic fraction of bosons occupies *one* quantum state
- This one state extends over the whole system, and basically corresponds to the superfluid
- Caveat: BEC as described really applies to non-interacting bosons. He IV atoms are very strongly interacting. But still the BEC starting point is qualitatively good. And we mostly just need the idea.

Landau-type theory

- Idea: the key property of the superfluid is a macroscopically occupied quantum state. We describe it with a *macroscopic wavefunction*

$$\psi(r) = \sqrt{n_s(r)}e^{i\theta(r)}$$

- Here n_s is the *superfluid density* and θ is the phase of the superfluid
- In equilibrium n_s is constant and so is θ. But there may be very stable non-equilibrium states where they vary : states with flow!

Free energy

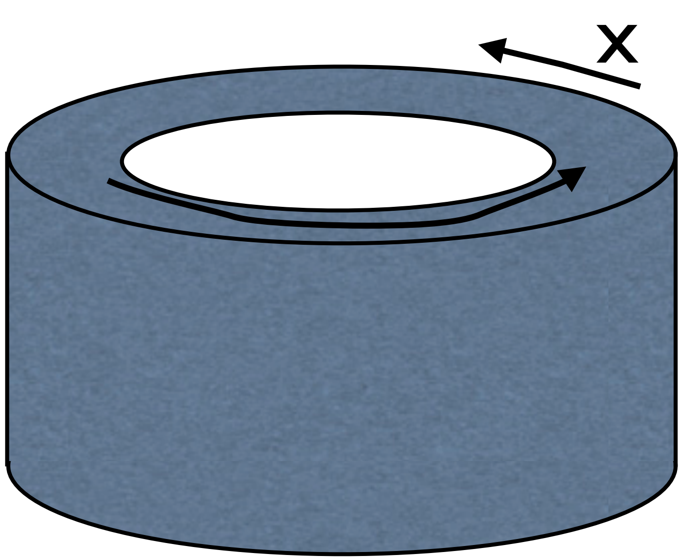
- We assume that the free energy of the system (which tends to its minimum in equilibrium) can be written in terms of n_s and θ:

$$F = \int d^3r \left[\frac{c}{2}(n_s - \overline{n}_s)^2 + \frac{\hbar^2 n_s}{2m}(\nabla\theta)^2 \right]$$

- Understand second term by noting that $\theta = \mathbf{k} \cdot \mathbf{r}$ is like putting all the "condensed" atoms into a state with momentum $\hbar\mathbf{k}$
- Superfluid velocity $\mathbf{v}_s = \frac{\hbar}{m}\nabla\theta$

Persistent current

- Obvious free energy is minimized by v_s=0. But suppose we set up a superflow?



narrow cylinder of circumference L

$$v_s^x = \frac{\hbar}{m}\partial_x\theta \quad \Rightarrow \quad \theta = \frac{mv_s}{\hbar}x$$

single valued wavefunction

$$\theta(L) - \theta(0) = \frac{mv_s}{\hbar}L = 2\pi N$$

"quantized circulation"

$$v_s = \frac{h}{mL}N$$

it is very hard for circulation to decay because it cannot change continuously!

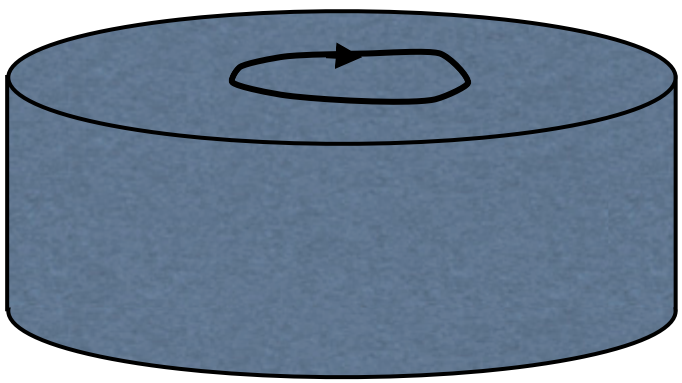
Vortices

- What happens if you rotate a bucket (not an annulus) above T_c, then cool it, then stop the bucket?
  - above T_c fluid has friction so will rotate
  - below T_c, has to keep rotating by conservation of angular momentum
- But if there is no center of the bucket, then there is a problem

Vortices

- Stoke's theorem

$$\oint \nabla\theta \cdot dr = \int \nabla \times \nabla\theta \cdot dA = 0$$

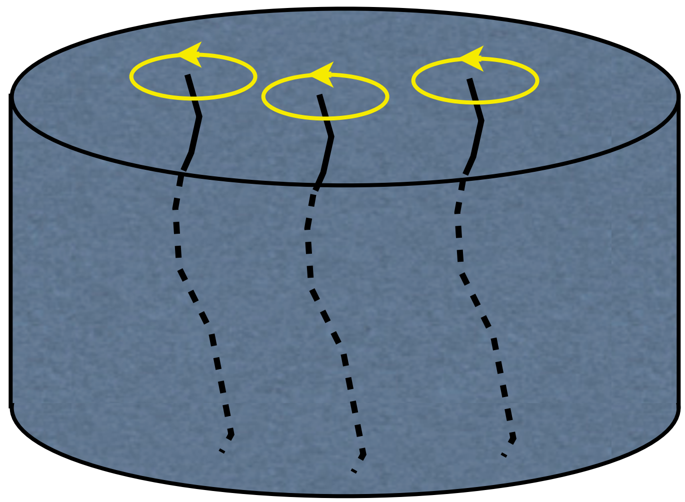


no circulation?

- In fact, circulation exists because *vortices* form. These are places in the liquid where $n_s \rightarrow 0$, and so θ is not well-defined!

Vortices

$$\theta \to \theta + 2\pi$$



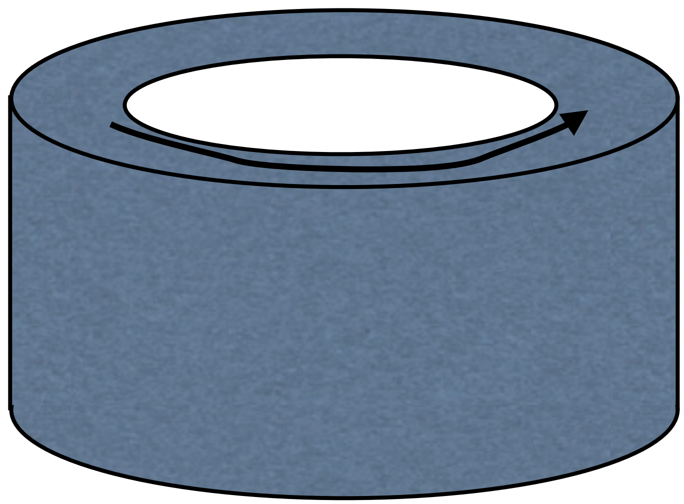
an array of vortices simulates rigid rotation of the fluid

$$\oint \nabla\theta \cdot dr = 2\pi$$ around a single vortex

- Vortex must escape the system - moving into the wall - to lower the circulation
- They will escape, because the vortex costs free energy - especially the core where $n_s = 0$, but also the $v_s \neq 0$ outside

Persistent current

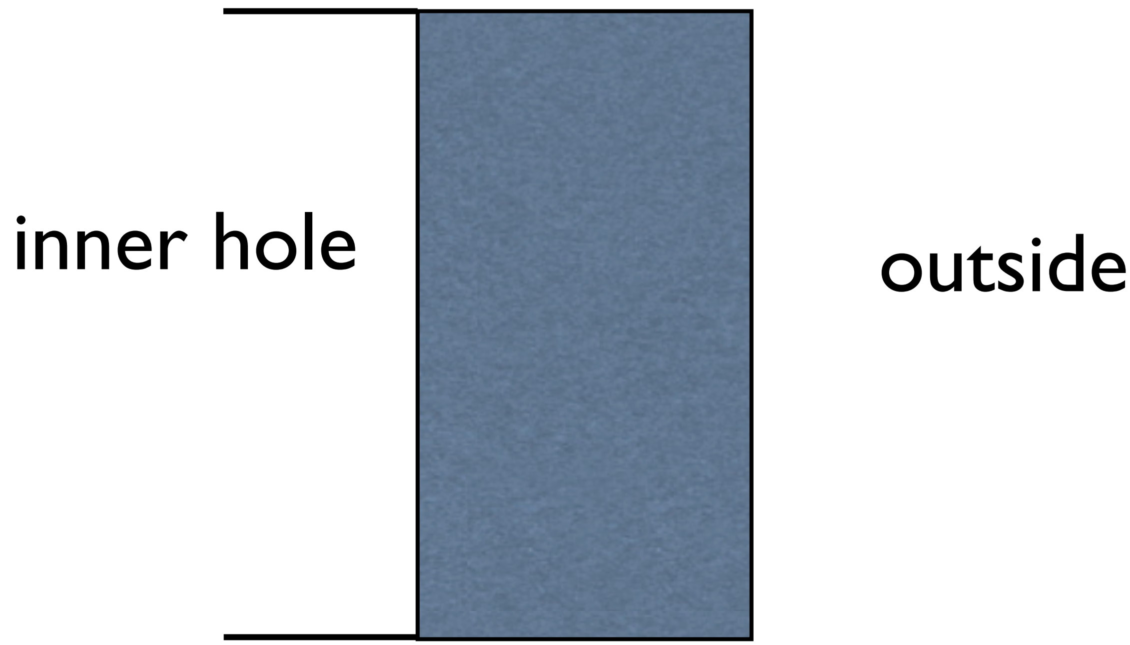
- In an annulus, one has a "giant vortex" in the hole: no core energy



- To decay, the circulation must escape *one vortex at a time*

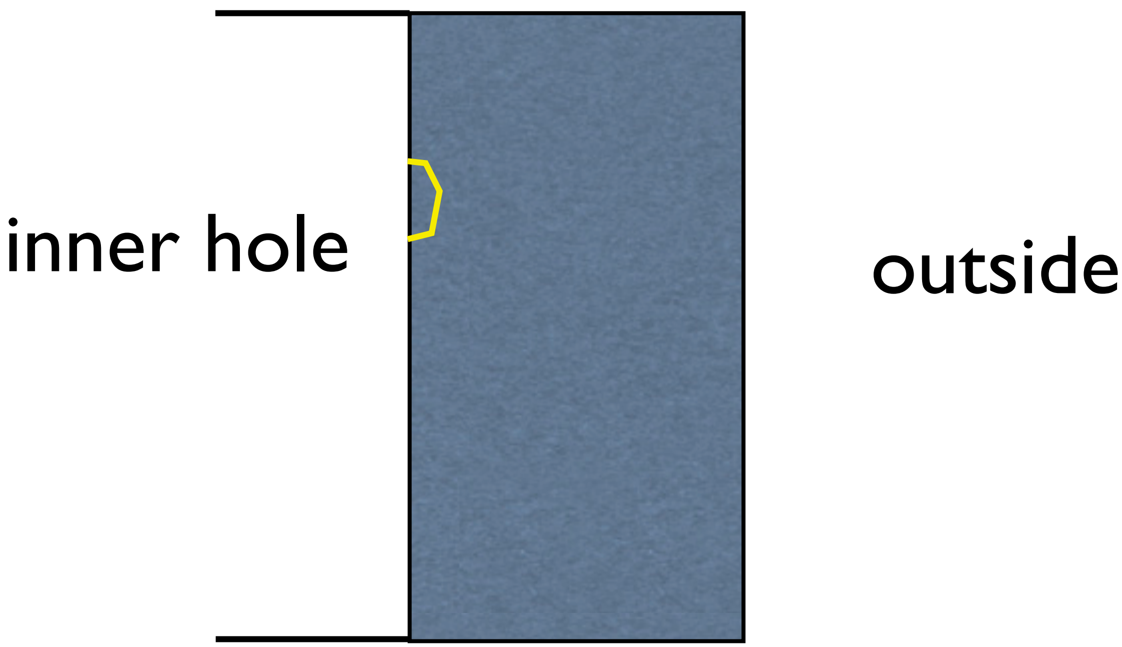
Vortex escape

- radial cut of annulus



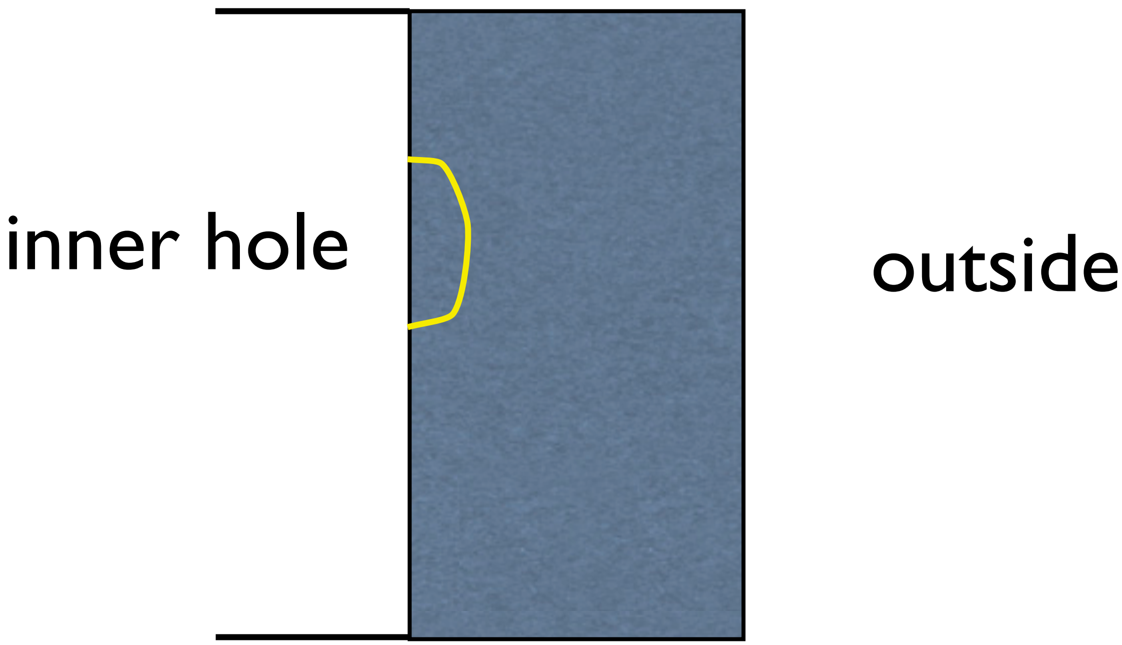
Vortex escape

- radial cut of annulus



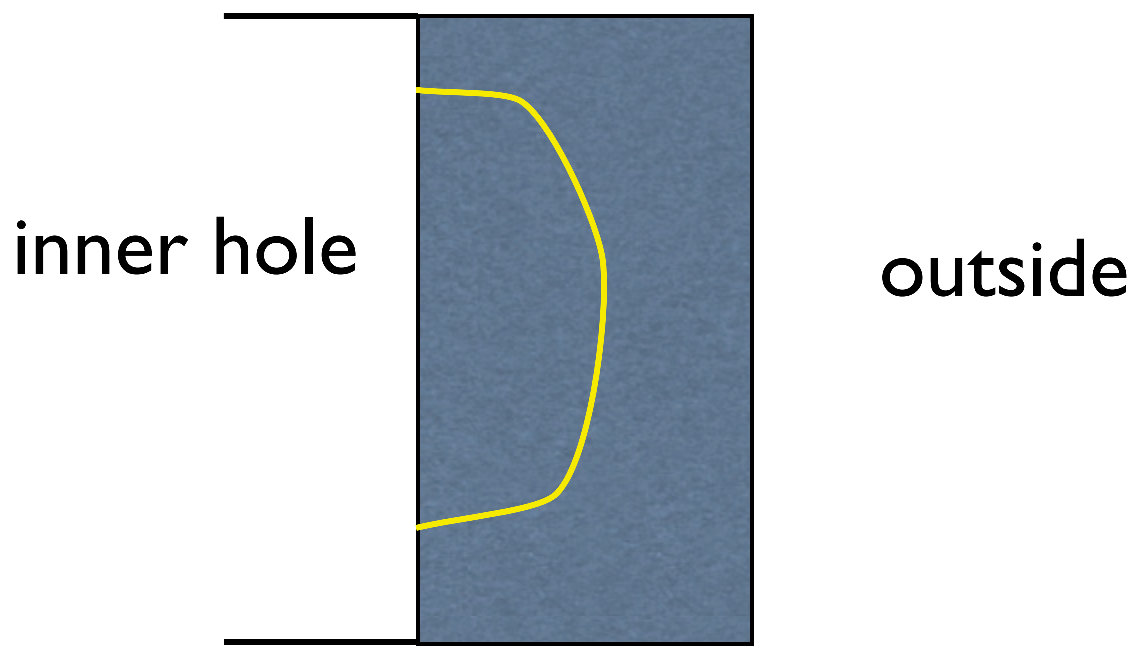
Vortex escape

- radial cut of annulus



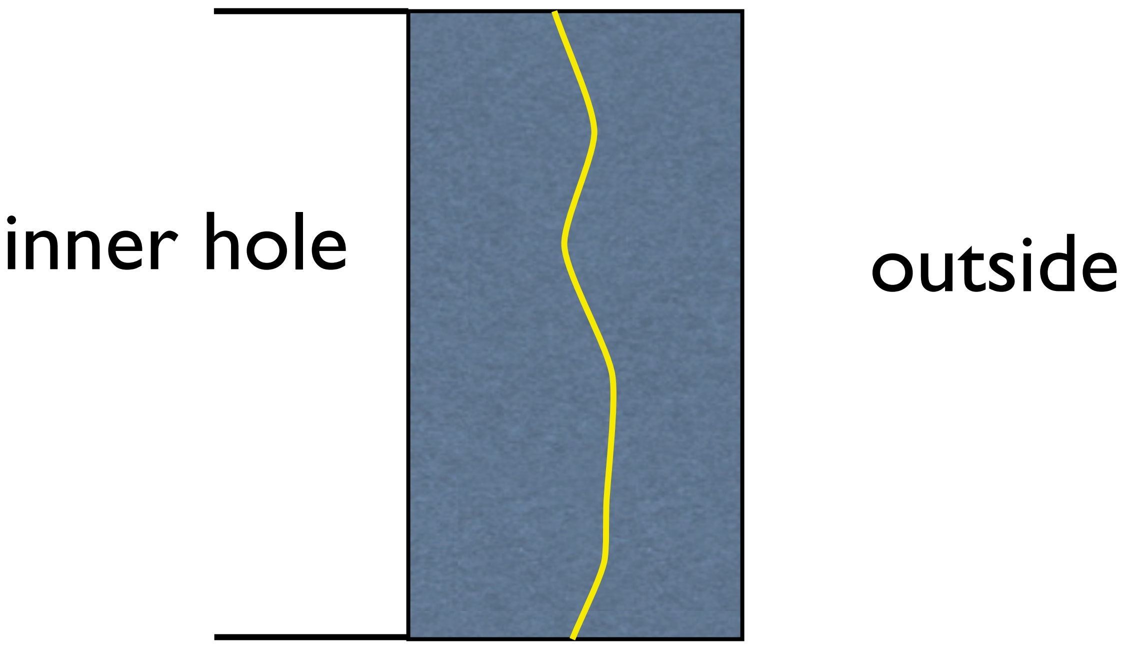
Vortex escape

- radial cut of annulus



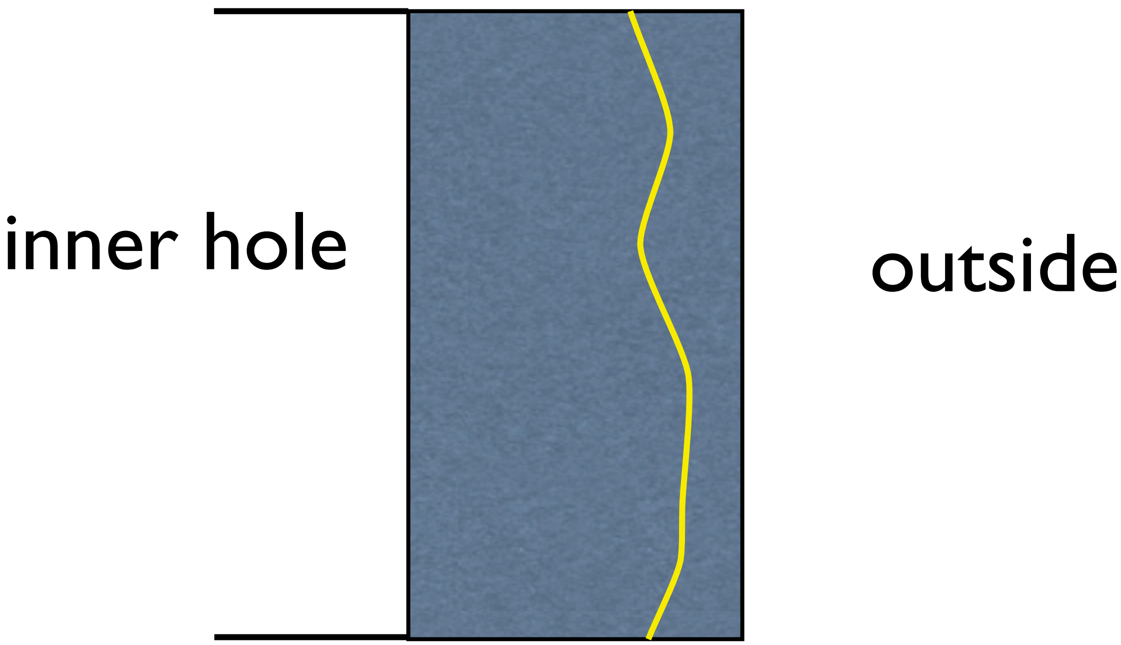
Vortex escape

- radial cut of annulus



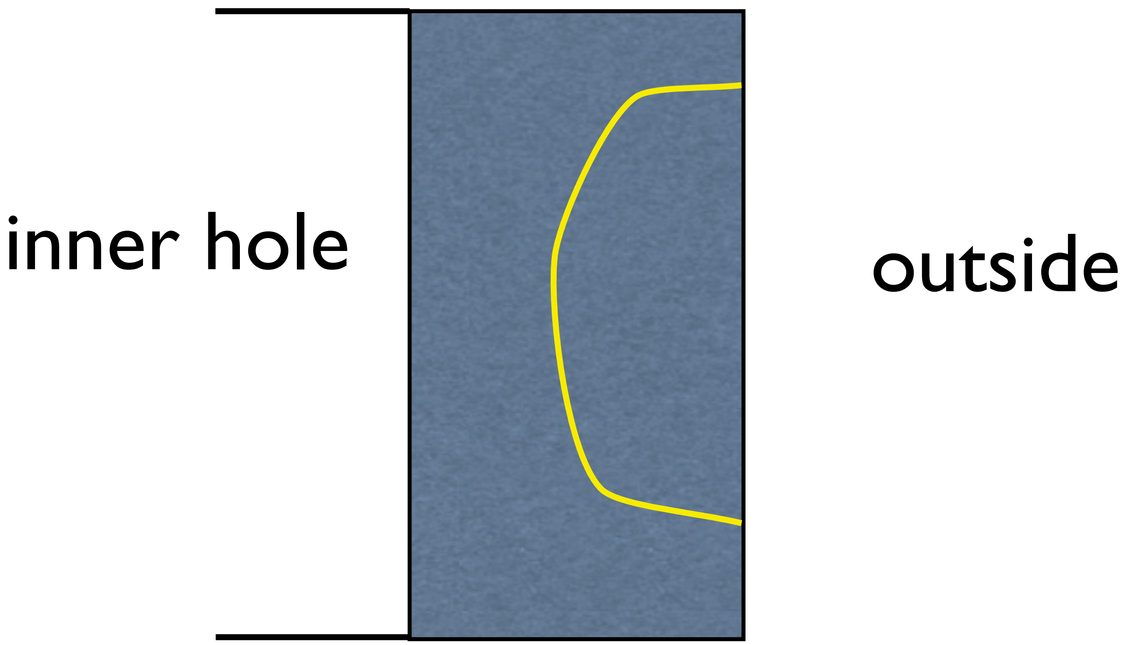
Vortex escape

- radial cut of annulus



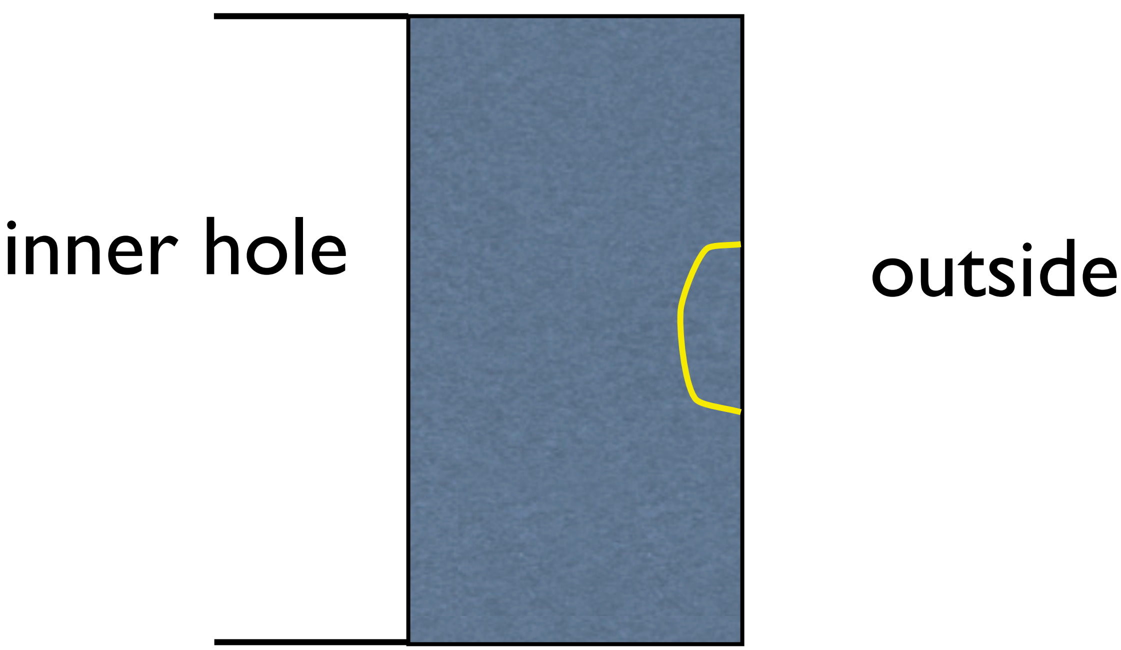
Vortex escape

- radial cut of annulus



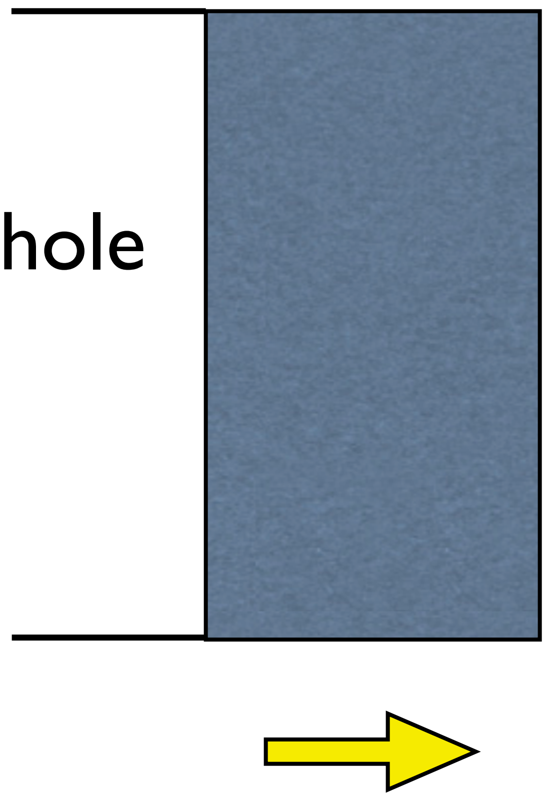
Vortex escape

- radial cut of annulus



Vortex escape

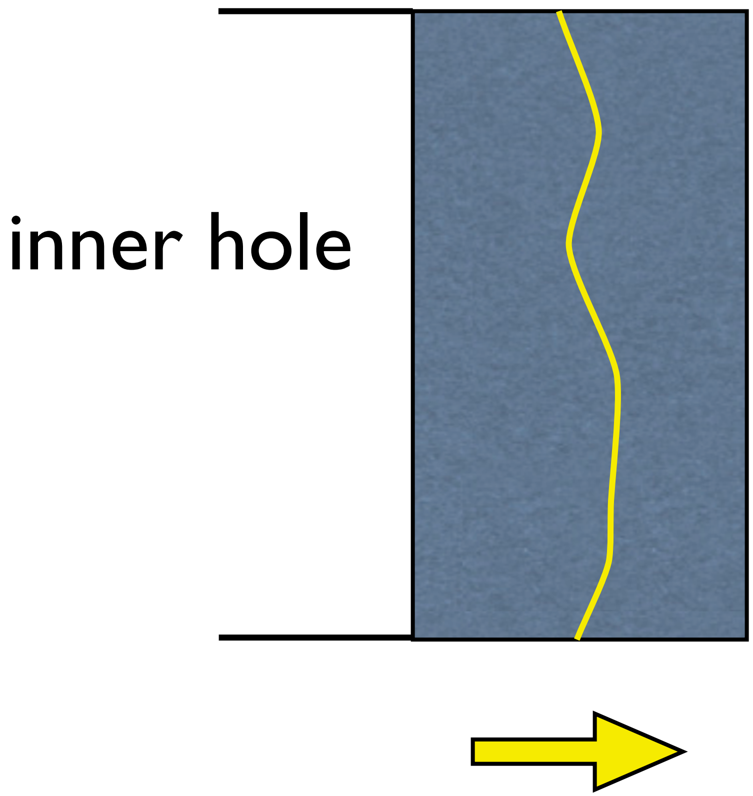
- radial cut of annulus



after this process, one quantum of circulation has been removed

Vortex escape

- radial cut of annulus

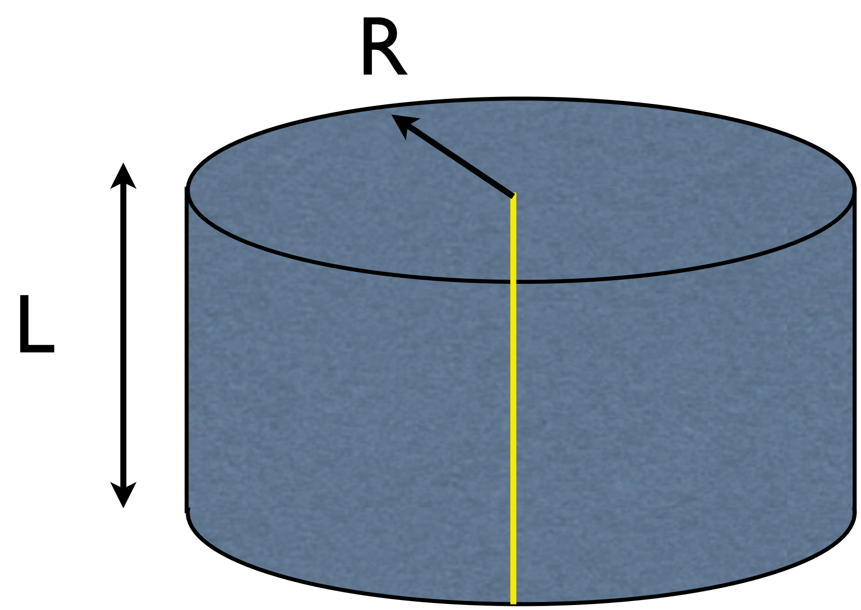


the intermediate state contains a long vortex line, which costs free energy. This free energy barrier requires thermal activation to overcome

after this process, one quantum of circulation has been removed

Vortex free energy

- Crude estimate $$F = \int d^3r \, \frac{\hbar^2 n_s}{2m} (\nabla\theta)^2 \qquad |\nabla\theta| = 1/r$$



$$F \approx L \int_a^R dr 2\pi r \frac{\hbar^2 n_s}{2m} \left(\frac{1}{r}\right)^2$$

$$\approx \frac{\pi \hbar^2 n_s L}{m} \ln(R/a)$$

even neglecting the log, this is of order 0.1K/Å

rate of thermal activation ~ $\exp(-F/k_B T)$

since T < 2.2K, this is negligible even for sub-micron dimensions

Second sound

- How does superfluid helium carry heat so quickly?
- Usually, heat diffuses if there is no fluid flow
- But superfluid helium behaves like *two fluids*: a normal and superfluid part
- Second sound = opposite flow of superfluid and normal fluids: no net mass flow but net heat flow
  - This is because the normal fluid convects heat, but the superfluid has *zero entropy* because it is a *single quantum state!*