Meissner effect

- To estimate the *critical field*, we need to compare the Gibbs free energy

$$G = \int d^3r \left[\frac{n_s e^2}{2m}|\mathbf{A}|^2 + a(n_s - n_s^{\mathrm{eq}})^2 + \frac{|\mathbf{B}|^2}{2\mu_0} - \mathbf{H}\cdot\mathbf{B}\right]$$

- In the SC state, $n_s = n_s^{eq}$, A=B=0

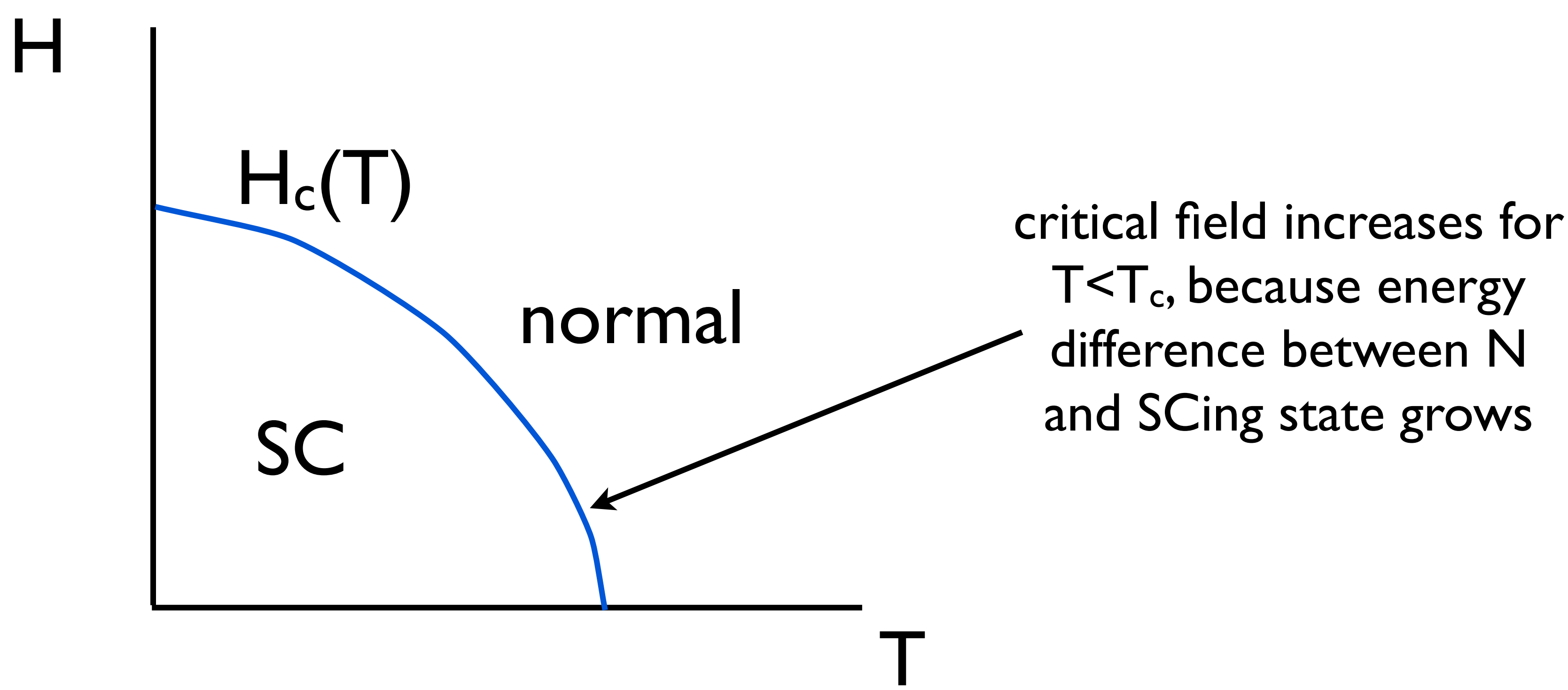
$$G_{sc} = 0$$

- In the normal state, $n_s=0$, $B=\mu_0 H$

$$G_n = V\left[a\left(n_s^{\mathrm{eq}}\right)^2 - \frac{\mu_0}{2}H^2\right]$$

- Equality $G_{sc}=G_n$ defines the critical field H_c

Meissner effect



This describes so-called “type I” superconductors

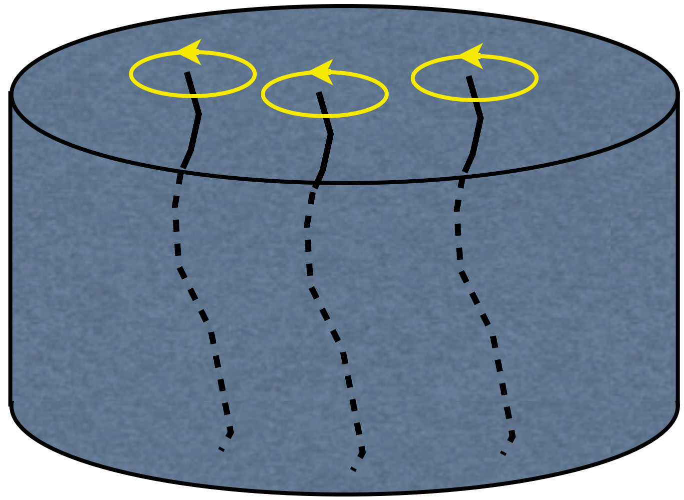
Some superconductors are “type II” and have a different phase diagram

Vortices

- In the previous, we assumed that the system had to be uniform and homogeneous
- It turns out that sometimes a non-uniform state is favored -- a collection of vortices
- Vortices are like those in superfluid helium, except that the "fluid" that is flowing is charged

Vortices

$$\theta \to \theta + 2\pi$$



$$\oint \nabla\theta \cdot dr = 2\pi$$

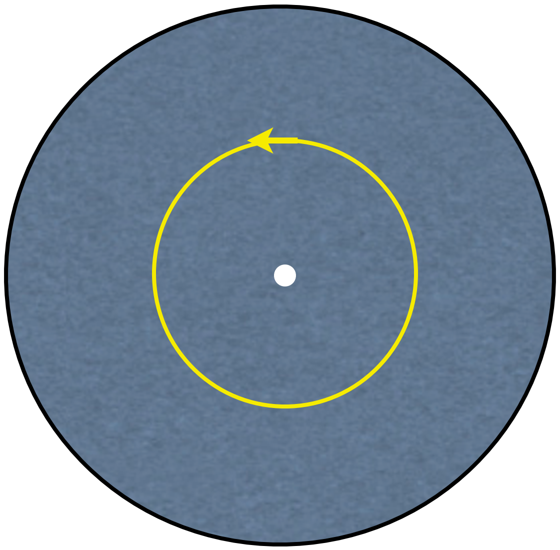
around a single vortex

- Free energy?

$$F = \int d^3r \left[\frac{n_s}{8m}|\hbar\nabla\theta + 2e\mathbf{A}|^2 + a(n_s - n_s^{\mathrm{eq}})^2 + \frac{B^2}{2\mu_0}\right]$$

- Minimized for $\mathbf{A} = \frac{\hbar}{2e}\nabla\theta \quad B = 0 \quad n_s = n_s^{\mathrm{eq}}$

Vortices



$$\mathbf{A} = \frac{\hbar}{2e}\nabla\theta$$

$$\mathbf{B} = \nabla \times \mathbf{A} = 0$$

$$\oint \mathbf{A}\cdot d\ell = \frac{\hbar}{2e}2\pi = \frac{h}{2e} = \varphi_0$$

- This implies *flux quantization*
- Note that it is quantized in units of *half* the flux quantum we saw in the IQHE
- This is directly related to the fact that Cooper *pairs* are condensed.

Vortices

- Apparent contradiction:

$$\mathbf{B} = \nabla \times \mathbf{A} = 0 \qquad \oint \mathbf{A} \cdot d\ell = \iint \mathbf{B} \cdot d\hat{\mathbf{z}} = \varphi_0$$

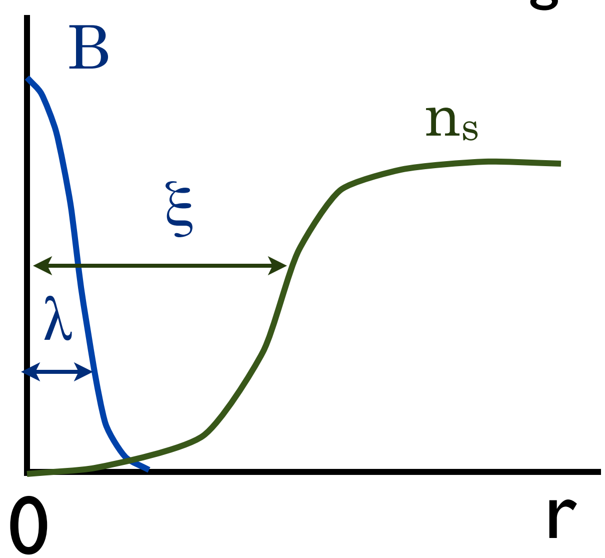
- This would seem to imply that

$$B_z = \varphi_0 \delta(x) \delta(y)$$

- We can have A ~ $\nabla\theta$ only *far* from the vortex core
    - So in reality the magnetic field is spread out
    - And in addition $n_s \rightarrow 0$ at the vortex core

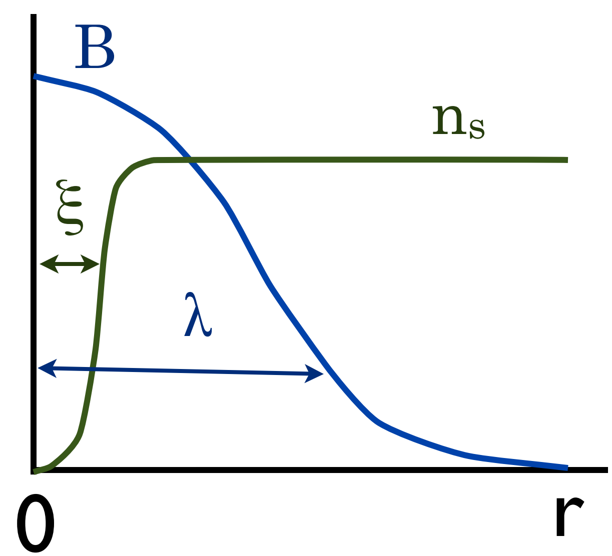
Vortices

- Flux is spread out over radius λ
- Condensed is depleted over radius ξ, called the *coherence length*



$\xi >> \lambda$

type I

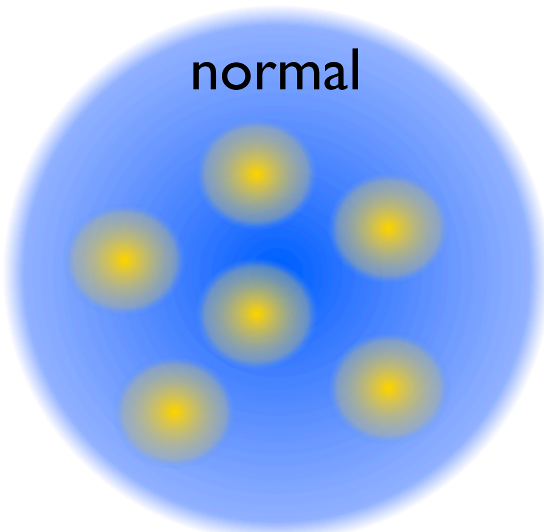


$\xi << \lambda$

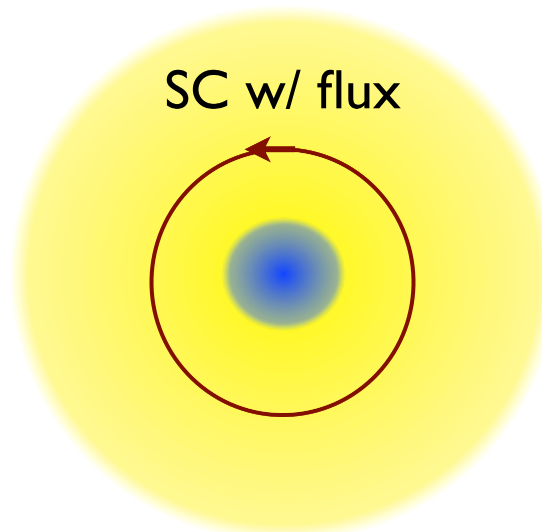
type II

Type I versus type II

type I



type II

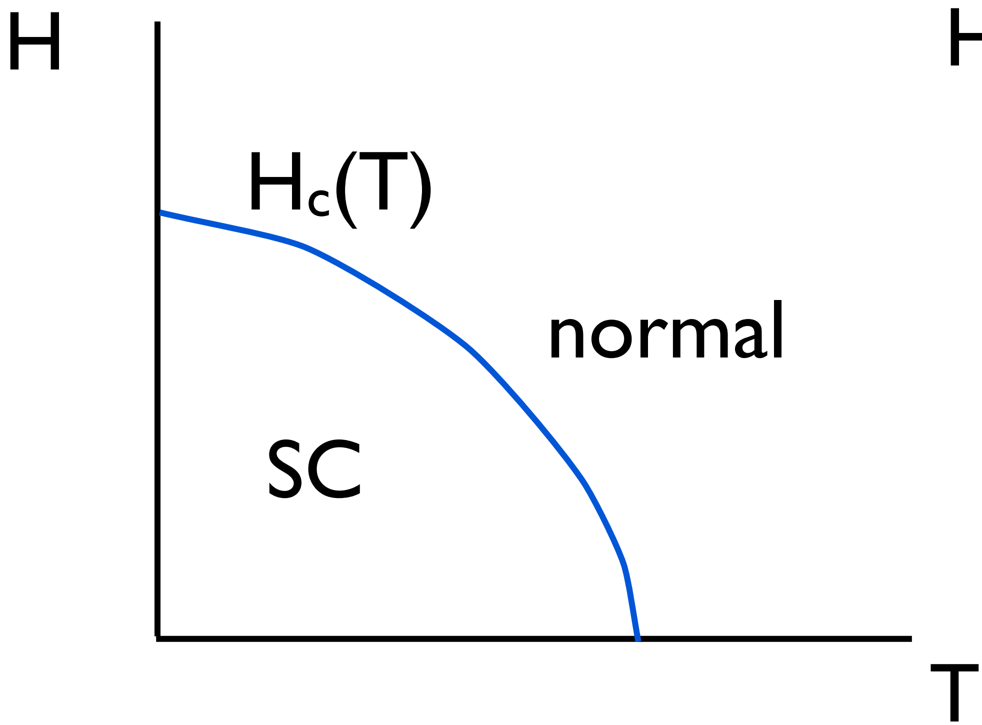


$$F = \int d^3r\,[\frac{n_s}{8m}|\hbar\nabla\theta + 2e\mathbf{A}|^2 + a(n_s - n_s^{\mathrm{eq}})^2 + \frac{B^2}{2\mu_0}]$$

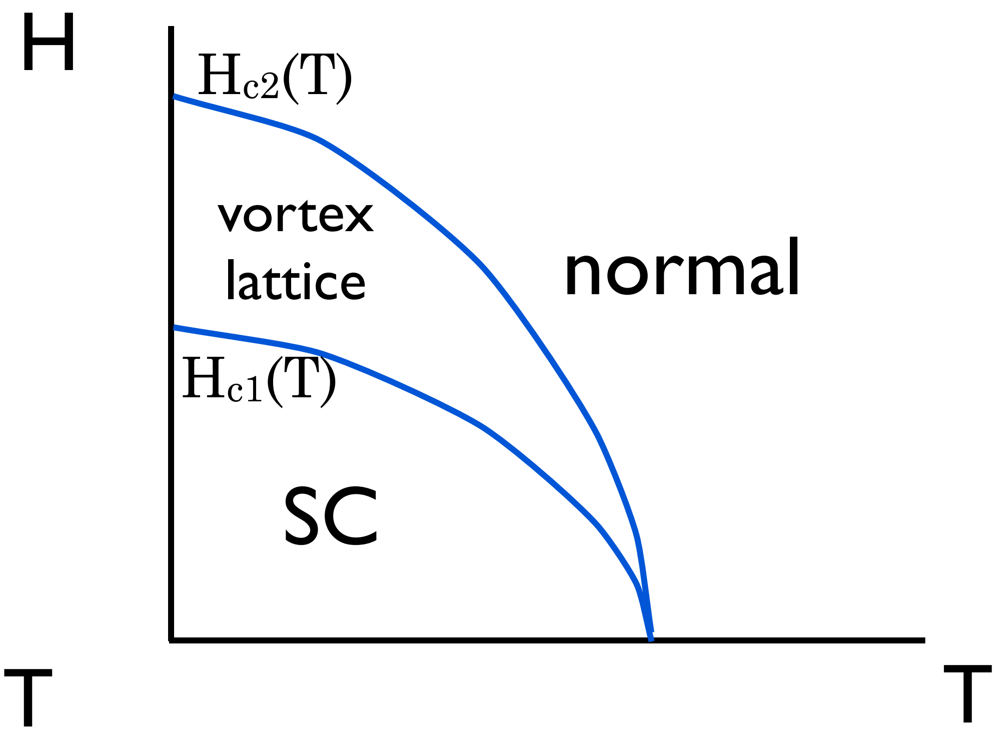
no cost to add more flux inside core: flux clumps together and enters system all at once

additional flux costs more energy.

Phase diagrams



Type I



Type II

$$\lambda/\xi > 1/\sqrt{2}$$

Quasiparticles

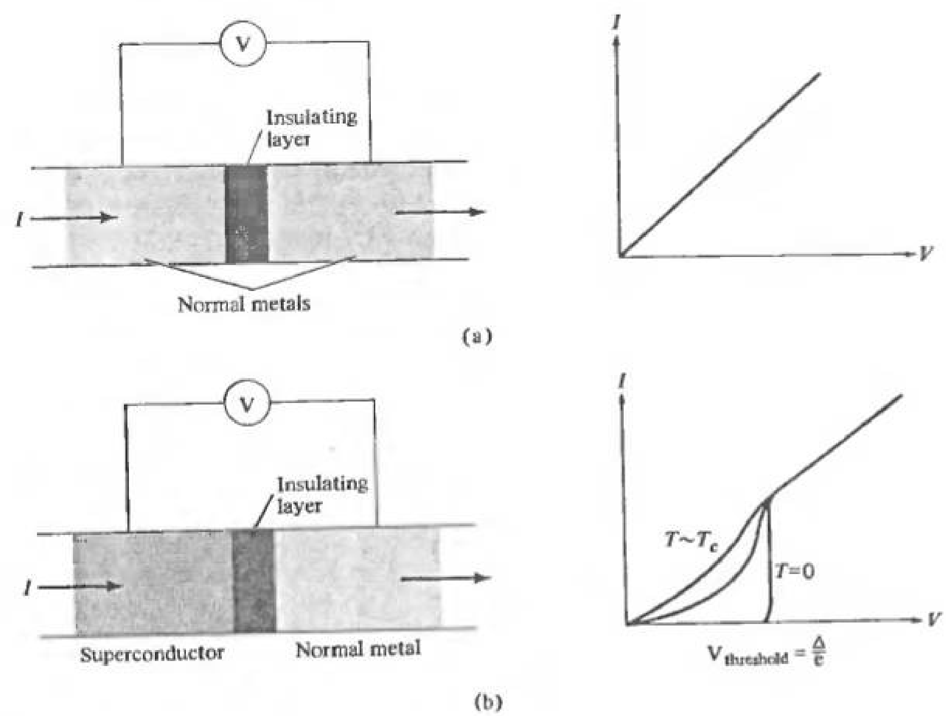
- In superfluid He, it is the elementary boson - the helium atom - which condenses.
- But in a superconductor, only pairs condense. We can still ask about individual *unpaired electrons*
- These are still fermions, and so are obviously not condensed
- In fact, since they are bound, it costs a non-zero energy to “break” a pair and create such “quasiparticles”. This is called the *gap*.

Quasiparticles

- Many experiments probe individual quasiparticles:
  - Tunneling
  - Photoemission
  - Thermal conductivity
  - Optics
  - ...

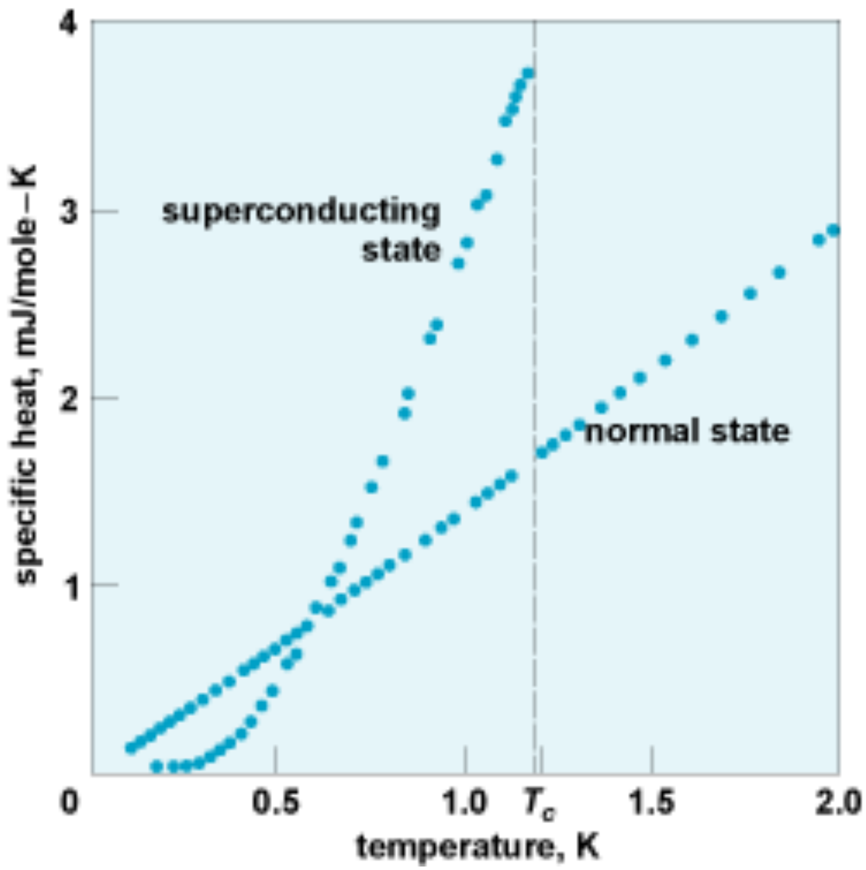
Tunneling

- Measures available density of states for quasiparticles



Specific heat

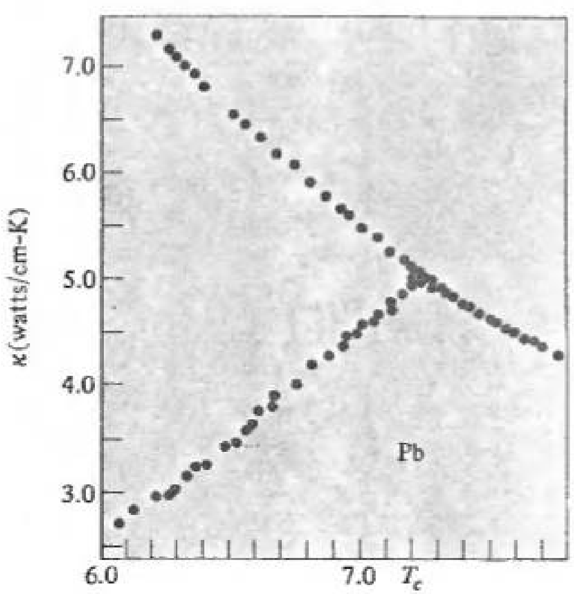
- Typically activated, ~ $e^{-\Delta/kT}$



Thermal conductivity

- Superconductors become thermally insulating. Note contrast to superfluids which have *ballistic* heat conduction. Due to fact that normal electrons diffuse instead of convecting

Figure 34.2
The thermal conductivity of lead. Below T_c the lower curve gives the thermal conductivity in the superconducting state, and the upper curve, in the normal state. The normal sample is produced below T_c by application of a magnetic field, which is assumed otherwise to have no appreciable affect on the thermal conductivity. (Reproduced by permission of the National Research Council of Canada from J. H. P. Watson and G. M. Graham, *Can. J. Phys.* **41**, 1738 (1963).)



BCS theory

Bardeen, Cooper, Schrieffer, 1957

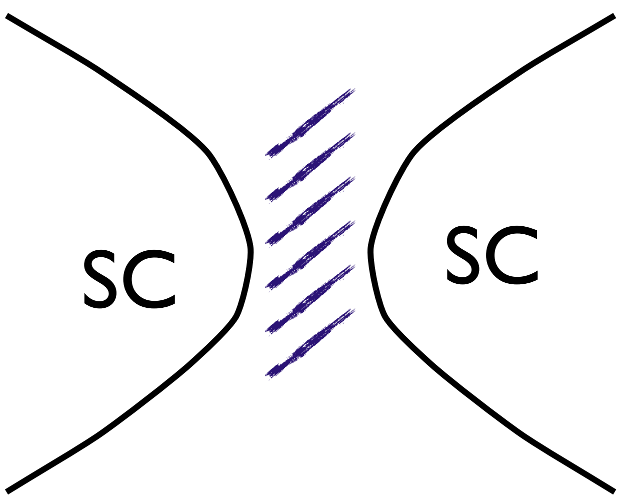
- For conventional superconductors, there is a quantitative theory of the mechanism, which describes how electrons pair and describes the quasiparticles
- This relies on the fact that $\xi >> \lambda_F$ in those materials, which means the pairs are "large" and highly overlapping
- This enables construction of a "mean field theory" - we will see an example of this later when we discuss magnetism

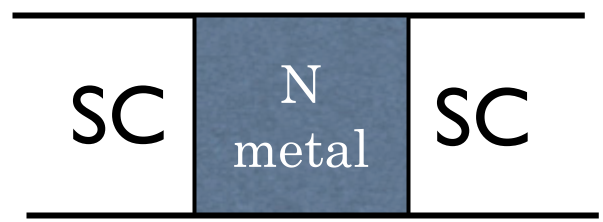
BCS theory

- Because $\xi \gg \lambda_F$, you cannot really think of Cooper pairs are tightly bound molecules
- Instead, onset of superconductivity is not so much BEC of Cooper pairs, but rather the point at which the pairs themselves form
- BCS theory predicts $\Delta(0) = 1.764kT_c$
  - as well as T dependence of gap, etc.

Josephson effects

- Occurs whenever two superconductors are connected by a "weak link", a narrow non-superconducting region





Josephson effects

- Josephson (1962): It is possible for a supercurrent to flow across the normal region - the "junction" - by tunneling
  - It is surprising this was noticed so late in the history of superconductivity
  - One of the rare theory-led discoveries (like TIs!)
- Josephson used microscopic BCS theory to derive this, but the effect is very general and can be understood without BCS theory.

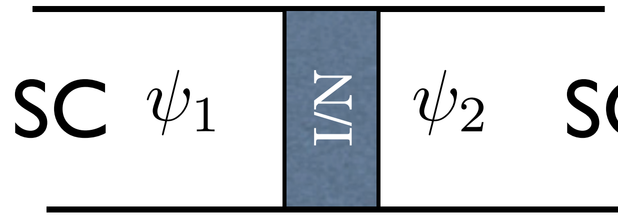
Free energy

$$\psi_i = |\psi| e^{i\theta_i}$$

- If electrons can move across the barrier (even a little), then they can transmit phase information from one SC to another
- Free energy will depend upon the phase difference

$$F \stackrel{?}{=} F_1 + F_2 - E_J \cos(\theta_1 - \theta_2)$$

Free energy



- This is not "gauge invariant" - it depends on how we choose our vector and scalar potentials for electromagnetism
- The gauge-invariant free energy is

$$F_J = -E_J \cos\left(\theta_1 - \theta_2 - \frac{2e}{\hbar}\int_1^2 \mathbf{A} \cdot d\ell\right)$$

= γ: gauge-invariant phase difference

Josephson relation

- Josephson realized there is a relation between the phase and the voltage

$$\partial_t \theta_i = -\frac{\epsilon_i}{\hbar} = \frac{2e}{\hbar}\varphi_i \qquad (\varphi = \text{scalar potential})$$

$$\partial_t \gamma = \frac{2e}{\hbar}(\varphi_1 - \varphi_2) - \frac{2e}{\hbar}\int_1^2 \partial_t \mathbf{A} \cdot d\mathbf{r}$$

$$\begin{aligned}\partial_t \gamma &= \frac{2e}{\hbar}\int_1^2 (-\nabla\varphi - \partial_t \mathbf{A}) \cdot d\mathbf{r} \\ &= \frac{2e}{\hbar}\int_1^2 \partial_t \mathbf{E} \cdot d\mathbf{r} = \frac{2e}{\hbar} V\end{aligned}$$

Current

- Consider work done by small change of phase:

$$dF = IV\,dt = I\frac{\hbar}{2e}d\gamma$$

- But

$$dF = d(-E_J\cos\gamma) = E_J\sin\gamma d\gamma$$

$$\boxed{I = \frac{2eE_J}{\hbar}\sin\gamma}$$

- Hence a constant phase can produce a supercurrent, with zero voltage, for $I<I_c$, with critical current $$I_c = \frac{2eE_J}{\hbar}$$

Size of critical current

- The amount of current a JJ can carry is obviously dependent upon the junction
- Natural to expect that I_c is correlated with the conductance G in the normal state
- Ambegaokar/Baratoff formula (BCS theory):

$$I_c R_n = \frac{\pi\Delta}{2e} \tanh\left(\frac{\Delta}{2kT}\right)$$

dimensions make sense!

Consequences

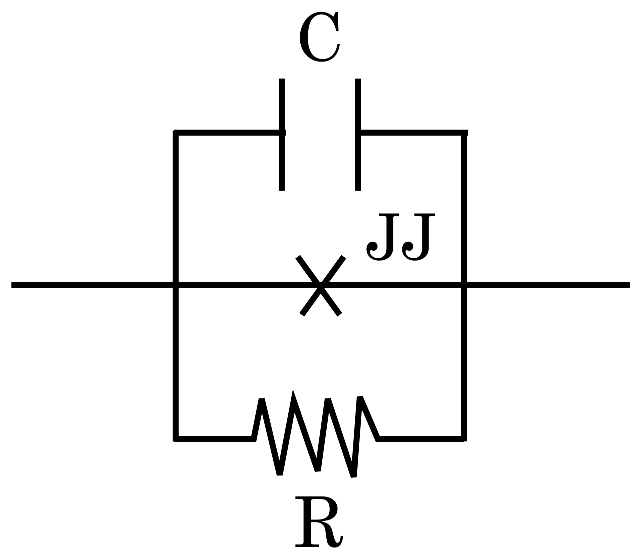
- Zero-bias (dissipationless) current
- AC Josephson effect: a voltage induces an oscillating current

Josephson frequency

$$\gamma = \frac{2eVt}{\hbar} \quad \Rightarrow \quad I(t) = I_c \sin \frac{2eV}{\hbar} t$$

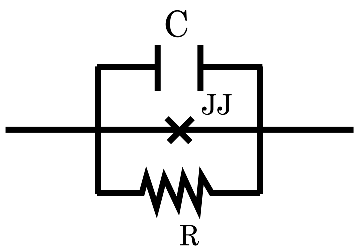
RCSJ model

- A simple model for the IV curve of a JJ



$$I = I_c \sin\gamma + \frac{V}{R} + C\frac{dV}{dt}$$

$$V = \frac{\hbar}{2e}\frac{d\gamma}{dt}$$

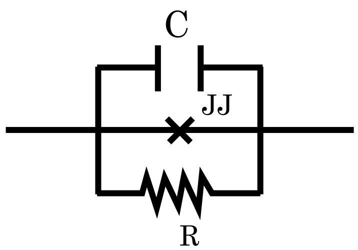


RCSJ model

- Same equation as a *pendulum* or particle in a *tilted washboard* potential

$$\frac{d^2\gamma}{dt^2} + \frac{1}{RC}\frac{d\gamma}{dt} + \frac{2eI_c}{\hbar C}\sin\gamma = \frac{2eI}{\hbar C}$$

- $I<I_c$: constant phase: V=0
- $I>I_c$: pendulum spins: non-zero average voltage



RCSJ model

- Consider over-damped limit $\frac{1}{RC} \gg \sqrt{\frac{2eI_c}{\hbar C}}$

$$\cancel{\frac{d^2\gamma}{dt^2}} + \frac{1}{RC}\frac{d\gamma}{dt} + \frac{2eI_c}{\hbar C}\sin\gamma = \frac{2eI}{\hbar C}$$

$$\frac{d\gamma}{dt} = \frac{2eI_cR}{\hbar}\left(\frac{I}{I_c} - \sin\gamma\right) > 0$$

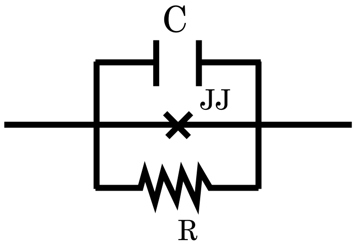
$$\int \frac{d\gamma}{I/I_c - \sin\gamma} = \int \frac{2eI_cR}{\hbar} dt = \frac{2eI_cR}{\hbar} t$$

time for one cycle

$$t_{2\pi} = \frac{\hbar}{2eI_cR}\int_0^{2\pi} \frac{d\gamma}{I/I_c - \sin\gamma} = \frac{\hbar}{2eI_cR}\frac{2\pi}{\sqrt{(I/I_c)^2 - 1}}$$

DC voltage

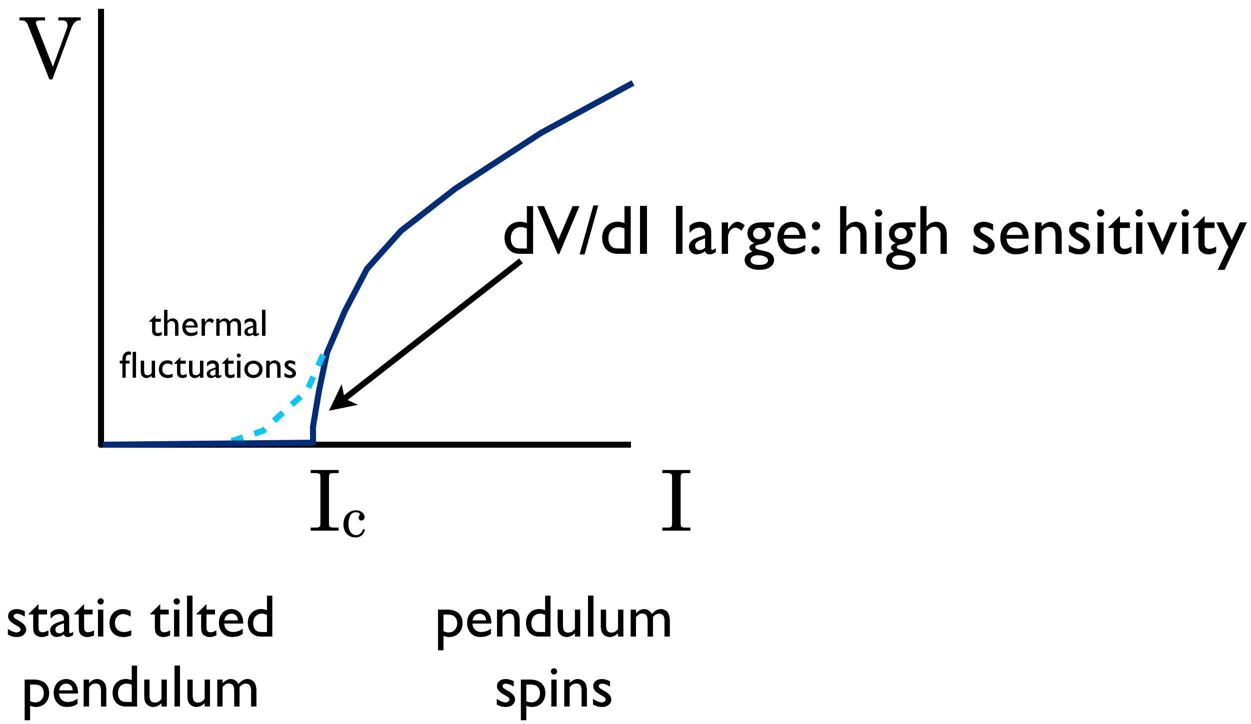
$$V = \frac{\hbar}{2e}\left\langle \frac{d\gamma}{dt} \right\rangle_{av} = \frac{\hbar}{2e}\frac{2\pi}{t_{2\pi}} = R\sqrt{I^2 - I_c^2}$$



RCSJ model

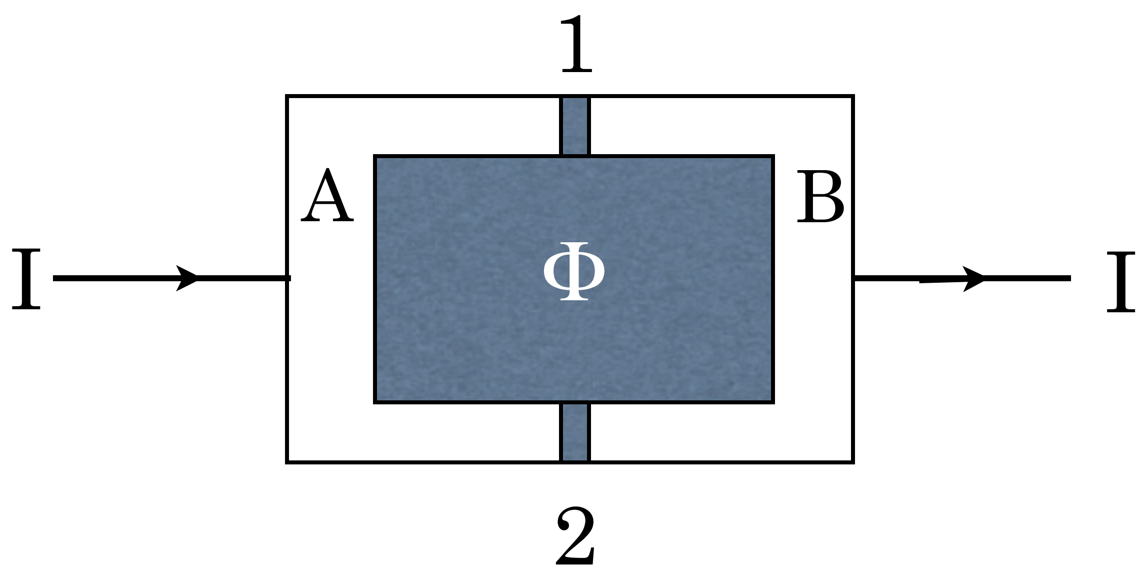
- Consider over-damped limit $\frac{1}{RC} \gg \sqrt{\frac{2eI_c}{\hbar C}}$

$$V = R\sqrt{I^2 - I_c^2}$$

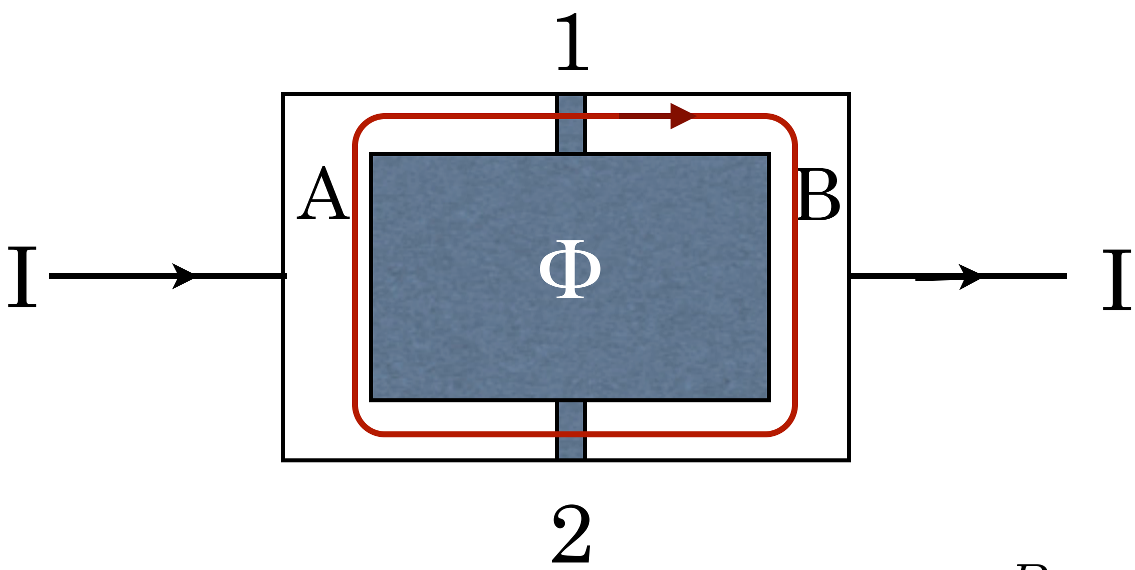


SQUIDs

- SQUID = Superconducting QUantum Interference Device.
- Many kinds of SQUIDs. Here just consider DC-SQUID, a sensitive magnetometer



dc SQUID



in SC, current is negligible, since only small currents can cross barriers.

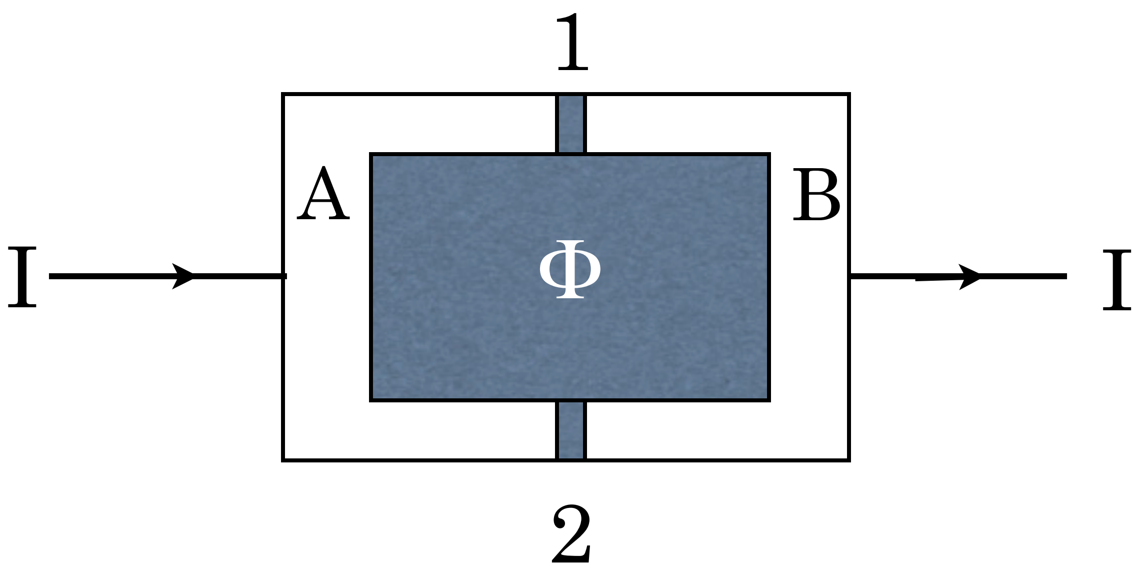
$$\mathbf{J} \propto \nabla\theta - \frac{2e}{\hbar}\mathbf{A} \approx 0$$

$$\oint \mathbf{A}\cdot d\ell = \oint_{\text{inside SC}} \mathbf{A}\cdot d\ell + \int_{A_1}^{B_1} \mathbf{A}\cdot d\ell + \int_{B_2}^{A_2} \mathbf{A}\cdot d\ell$$

$$\Phi = \frac{\hbar}{2e}\oint_{\text{inside SC}} \nabla\theta\cdot d\ell + \int_{A_1}^{B_1} \mathbf{A}\cdot d\ell + \int_{B_2}^{A_2} \mathbf{A}\cdot d\ell$$

$$= \frac{\hbar}{2e}\left(\theta_{A_1} - \theta_{A_2} + \theta_{B_2} - \theta_{B_1}\right) + \int_{A_1}^{B_1} \mathbf{A}\cdot d\ell + \int_{B_2}^{A_2} \mathbf{A}\cdot d\ell = \frac{1}{2\pi}\frac{h}{2e}\left(\gamma_1 - \gamma_2\right)$$

dc SQUID

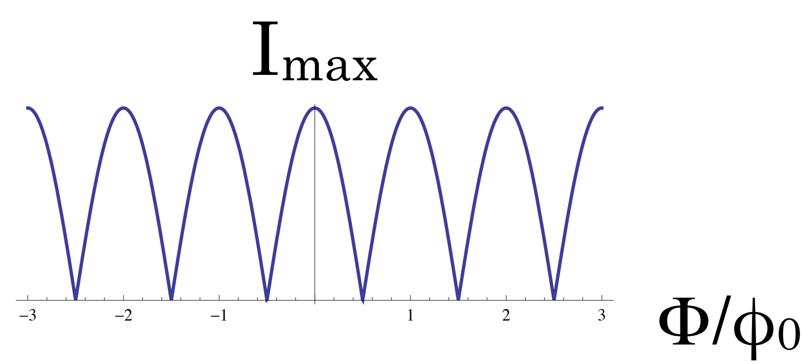


$$\gamma_1 - \gamma_2 = 2\pi \left(\frac{\Phi}{\varphi_0} \right)$$

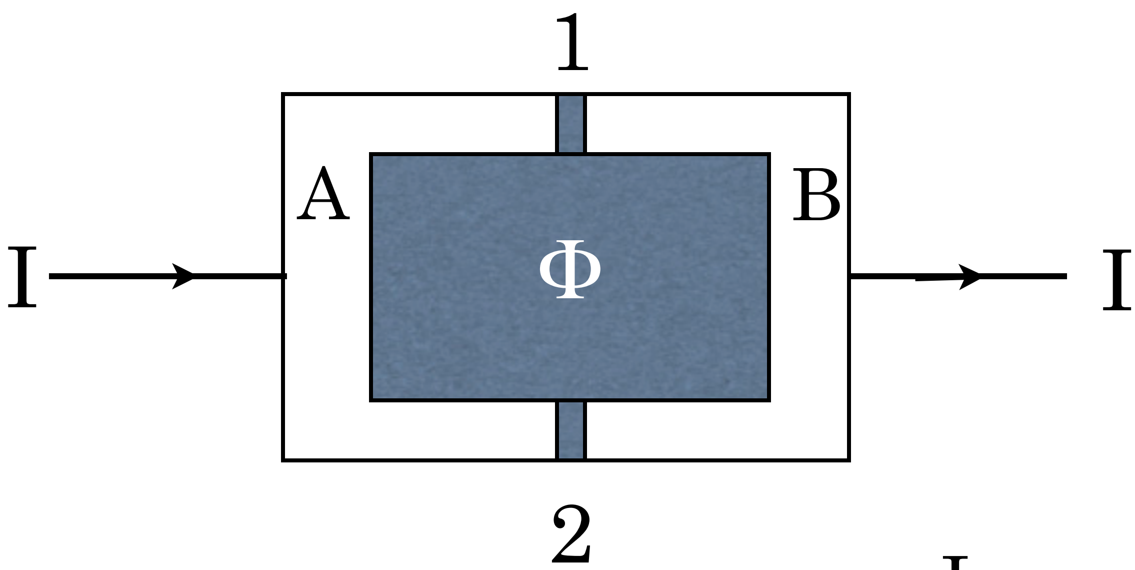
$$I = I_c \sin \gamma_1 + I_c \sin \gamma_2 \; = 2 I_c \cos \left(\frac{\gamma_1 - \gamma_2}{2} \right) \sin \left(\frac{\gamma_1 + \gamma_2}{2} \right)$$

$$= 2 I_c \cos \left(\frac{\pi \Phi}{\varphi_0} \right) \sin \left(\frac{\gamma_1 + \gamma_2}{2} \right)$$

$$I_{\max} = 2 I_c \left| \cos \left(\frac{\pi \Phi}{\varphi_0} \right) \right|$$

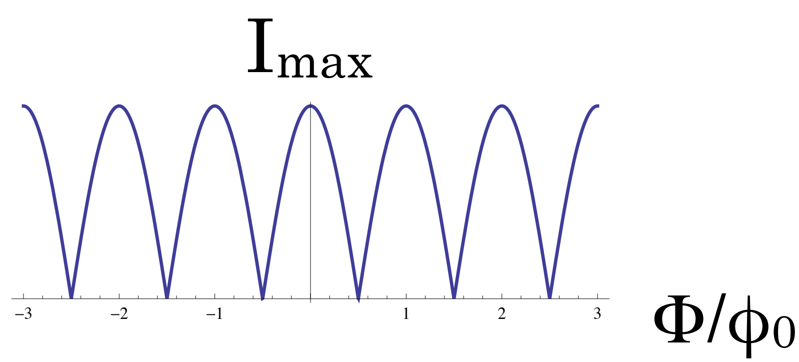


dc SQUID



$$\gamma_1 - \gamma_2 = 2\pi\left(\frac{\Phi}{\varphi_0}\right)$$

$$I_{\mathrm{max}} = 2I_c\left|\cos\left(\frac{\pi\Phi}{\varphi_0}\right)\right|$$



typically operate slightly above the max critical current, where voltage then varies rapidly with flux