Phase transition

- A lot happens at $T_c$

- Both $m(T)$ and $\chi(T)$ are non-analytic at $T_c$

- This is actually a sign of a phase transition
Quantum treatment

- So far, we treated the ferromagnet in mean field theory
- This is approximate. Usually qualitatively correct but not even always that.
- We can do better for the Heisenberg ferromagnet
- Goal: find the actual ground state and excitations
Quantum treatment

- Hamiltonian
  \[ H = -J \sum_{\langle ij \rangle} S_i \cdot S_j \]

  \[ = -J \sum_{\langle ij \rangle} \left( S_i^z S_j^z + \frac{1}{2} S_i^+ S_j^- + \frac{1}{2} S_i^- S_j^+ \right) \]

- Try the “obvious” ground state
  \[ |0\rangle = \prod_i |S_i^z = +S\rangle \]

  \[ H|0\rangle = -J \sum_{\langle ij \rangle} S^2 |0\rangle = -NJS^2 \frac{z}{2} |0\rangle \]
Quantum treatment

• Is it the ground state?

\[ H = -J \sum_{\langle ij \rangle} \left[ \left( \frac{S_i + S_j}{2} \right)^2 - S(S + 1) \right] \]

• Since \( 0 \leq |S_i + S_j| \leq 2S \) for two spins

\[ E \geq -J \sum_{\langle ij \rangle} \left[ \frac{2S(2S + 1)}{2} - S(S + 1) \right] = -JS^2 \frac{Nz}{2} \]

• Thus this is indeed a ground state!
Quantum treatment

- Excitations - lower the spin once

\[ |i\rangle = \frac{S_i^-}{\sqrt{2S}} \prod_j |S_j^z = S\rangle = |S_i^z = S - 1\rangle \prod_{j \neq i} |S_j^z = S\rangle \]

e.g.

\( S = 1/2 \)
Quantum treatment

• Hamiltonian

\[ H = -J \sum_{\langle ij \rangle} \left( S_i^z S_j^z + \frac{1}{2} S_i^+ S_j^- + \frac{1}{2} S_i^- S_j^+ \right) \equiv H_z + H_\pm \]

• zz terms

\[ H_z |i\rangle = (E_0 - Jz[S(S - 1) - S^2]) |i\rangle = (E_0 + JSz) |i\rangle \]
Quantum treatment

• +- terms

\[
H_\pm = -\frac{J}{2} \sum_{\langle jk \rangle} \left( S_j^+ S_k^- + S_j^- S_k^+ \right)
\]

\[
H_\pm |i\rangle = -\frac{J}{2} \sum_{\langle ji \rangle} S_j^- S_i^+ |i\rangle
\]

\[
= -\frac{J}{2} 2S \sum_{\langle ji \rangle} |j\rangle = -JS \sum_{\mu} |i + \mu\rangle
\]
Quantum treatment

• All together

\[ H|\rangle = (E_0 + J S z) |\rangle - J S \sum_\mu |\rangle + \mu \]

• Looks like a hopping Hamiltonian

• Bloch:

\[ |k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{i k \cdot r_i} |i\rangle \]

\[ H|k\rangle = \left( E_0 + J S z - J S \sum_\mu e^{i k \cdot e_\mu} \right) |k\rangle \]

\[ E_0 + \epsilon(k) \]
Quantum treatment

- These are “spin waves” or “magnons”

\[ \epsilon(k) = J S z - J S \sum_{\mu} e^{i k \cdot e_\mu} \]

\[ = 2 J S \left( d - \sum_{\alpha=1}^{d} \cos k_\alpha a \right) \]

\[ \approx J S a^2 k^2 \quad \text{quadratic for small } k \]

- Magnons are gapless
Magnons

• Magnons are gapless
• Energy vanishes as $k \to 0$
• This is because
  \[ |k = 0\rangle \propto S_{\text{TOT}}^-|0\rangle \]
• This is an example of a “Goldstone mode”
• “Goldstone’s theorem”: if $H$ has a continuous symmetry that is “broken” by the ground state, there will be a gapless mode
Magnons

- Spin wave/magnon can be regarded as a quantized precession wave of slightly tilted spins

- $\varepsilon \sim k^2$ behavior, a general property for ferromagnets, can be understood this way
Magnons

• Think of effective field due to other spins

\[ h(r) \approx c_0 m + c_1 \nabla^2 m + \cdots \]

• Local spins precess in this field

\[ \partial_t m = h \times m \approx c_1 \nabla^2 m \times m \]
Magnons

\[ \partial_t \mathbf{m} = \hbar \times \mathbf{m} \approx c_1 \nabla^2 \mathbf{m} \times \mathbf{m} \]

\[ \mathbf{m} = (m_x, m_y, \sqrt{m_0^2 - m_x^2 - m_y^2}) \approx (m_x, m_y, m_0) \]

\[ \partial_t \begin{pmatrix} m_x \\ m_y \end{pmatrix} = c_1 m_0 \begin{pmatrix} 0 & -k^2 \\ k^2 & 0 \end{pmatrix} \begin{pmatrix} m_x \\ m_y \end{pmatrix} \]

\[ \omega = \pm c_1 m_0 k^2 \]
Neutron scattering

- Neutron has a $S=1/2$ similar to an electron, and its own dipole moment, which interacts with magnetic dipoles in materials

\[
H_{d-d} = -\frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \mathbf{r})(\mathbf{m}' \cdot \mathbf{r}) - \mathbf{m} \cdot \mathbf{m}']
\]

- Consequently, a neutron can exchange energy and momentum with electronic spins

\[k_i, \omega_i \quad k = k_i - k_f, \omega = \omega_i - \omega_f \]
Neutron scattering

- Integrate over all energies: get total scattering
- Tool for detecting magnetic ordering in solids
- Resolve both momentum and energy of neutrons: measure spin waves

example of spin waves in Yb$_2$Ti$_2$O$_7$ (2012)

first measurement in an insulating FM: CrBr$_3$, 1971