Landau levels

- Simplest case: "free" 2d electrons in a magnetic field (applies to electrons in a semiconductor 2DEG)
- Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} + e\mathbf{A}\right)^2 \qquad\qquad \mathbf{A} = By\hat{x}$$

- Choose k_x eigenstate

$$\psi(x, y) = e^{ik_x x} Y(y)$$

Landau levels

- One obtains

$$\frac{1}{2m}\left(-\hbar^2\frac{d^2}{dy^2}+(eB)^2\left(y-\frac{\hbar k_x}{eB}\right)^2\right)Y=\epsilon Y$$

- This is a 1d simple harmonic oscillator with a frequency and center

$$\omega_c=\frac{eB}{c} \qquad y_0=\frac{\hbar k_x}{eB}=k_x\ell^2 \qquad \ell=\sqrt{\frac{\hbar}{eB}}$$

cyclotron frequency

magnetic length

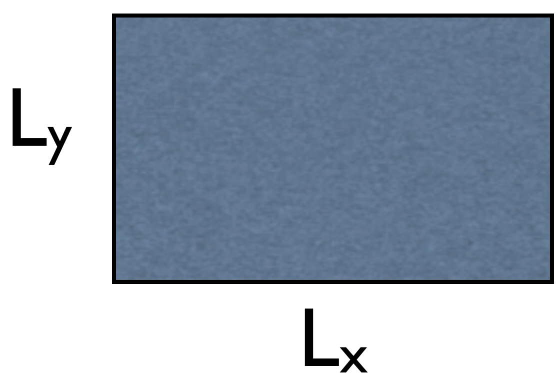
Landau levels

- Energy levels = Landau levels are

$$\epsilon_n = \hbar\omega_c(n + \tfrac{1}{2}) \qquad\qquad n = 0, 1, 2, \cdots$$

- Each is highly degenerate due to independence of energy on k_x

- How many?



$$k_x = \frac{2\pi}{L_x} i, \quad i = 0, 1, 2, \cdots$$

$$0 < y_0 = k_x \ell^2 < L_y$$

$$0 < i < \frac{L_x L_y}{2\pi\ell^2} \qquad N = \frac{A}{2\pi\ell^2}$$

Landau levels

- Degeneracy

$$N = \frac{A}{2\pi\ell^2} = AB\frac{e}{h} = \frac{\Phi}{\varphi}$$

- Flux quantum

$$\varphi = h/e \approx 4 \times 10^{-15} T \cdot m^2$$

- This is basically the number of minimal quantized cyclotron orbits which fit into the sample area

Dirac Landau Levels

- We saw that Schrödinger electrons form Landau levels with even spacing.
- It turns out Dirac electrons also form Landau levels but with different structure
- We can just follow the treatment in the graphene RMP

Dirac Landau levels

$$\begin{aligned} H &= v\vec{\sigma}\cdot(\vec{p}+e\vec{A}) \\ &= -i\hbar v\vec{\sigma}\cdot(\vec{\nabla}+i\frac{e}{\hbar}\vec{A}) \\ &= \hbar v\begin{pmatrix} 0 & -i\partial_x - \partial_y + \frac{eB}{\hbar}y \\ -i\partial_x + \partial_y + \frac{eB}{\hbar}y & 0 \end{pmatrix} \end{aligned}$$

$$H\psi = E\psi \qquad\qquad \psi(x,y) = e^{ik_x x}\phi(y)$$

$$\hbar v\begin{pmatrix} 0 & k_x - \partial_y + \frac{eB}{\hbar}y \\ k_x + \partial_y + \frac{eB}{\hbar}y & 0 \end{pmatrix}\phi(y) = E\phi(y)$$

Dirac LLs

$$\ell = \sqrt{\frac{\hbar}{eB}}$$

$$\frac{\hbar v}{\ell}\begin{pmatrix} 0 & k_x\ell - \partial_{y/\ell} + \frac{eB}{\hbar}y\ell \\ k_x\ell + \partial_{y/\ell} + \frac{eB}{\hbar}y\ell & 0 \end{pmatrix}\phi(y) = E\phi(y)$$

$$\frac{eB}{\hbar}y\ell \rightarrow y/\ell$$

$$\phi(y) = \Phi(y/\ell + k_x\ell) \qquad \omega_c = \sqrt{2}v/\ell$$

$$\hbar\omega_c\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\left(-\partial_\xi + \xi\right) \\ \frac{1}{\sqrt{2}}\left(\partial_\xi + \xi\right) & 0 \end{pmatrix}\Phi(\xi) = E\Phi(\xi)$$

Dirac LLs

$$\hbar\omega_c \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \Phi(\xi) = E\Phi(\xi)$$

$$a = \frac{1}{\sqrt{2}}(\partial_\xi + \xi)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(-\partial_\xi + \xi)$$

$$[a, a^\dagger] = 1 \qquad N = a^\dagger a$$

$$N|n\rangle = n|n\rangle \qquad a|0\rangle = 0$$

etc.

$$\Phi = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \Phi = \begin{pmatrix} 0 \\ a|0\rangle \end{pmatrix} = 0$$

Zero energy state: lives entirely on "A" sublattice

For the K' point it lives on the B sublattice

Dirac LLs

$$\hbar\omega_c \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \Phi(\xi) = E\Phi(\xi)$$

More general state $\Phi = \begin{pmatrix} |n\rangle \\ c|n-1\rangle \end{pmatrix}$

$$\hbar\omega_c \begin{pmatrix} a^\dagger c|n-1\rangle \\ a|n\rangle \end{pmatrix} = E \begin{pmatrix} |n\rangle \\ c|n-1\rangle \end{pmatrix}$$

$$\hbar\omega_c \begin{pmatrix} c\sqrt{n}|n\rangle \\ \sqrt{n}|n-1\rangle \end{pmatrix} = E \begin{pmatrix} |n\rangle \\ c|n-1\rangle \end{pmatrix}$$

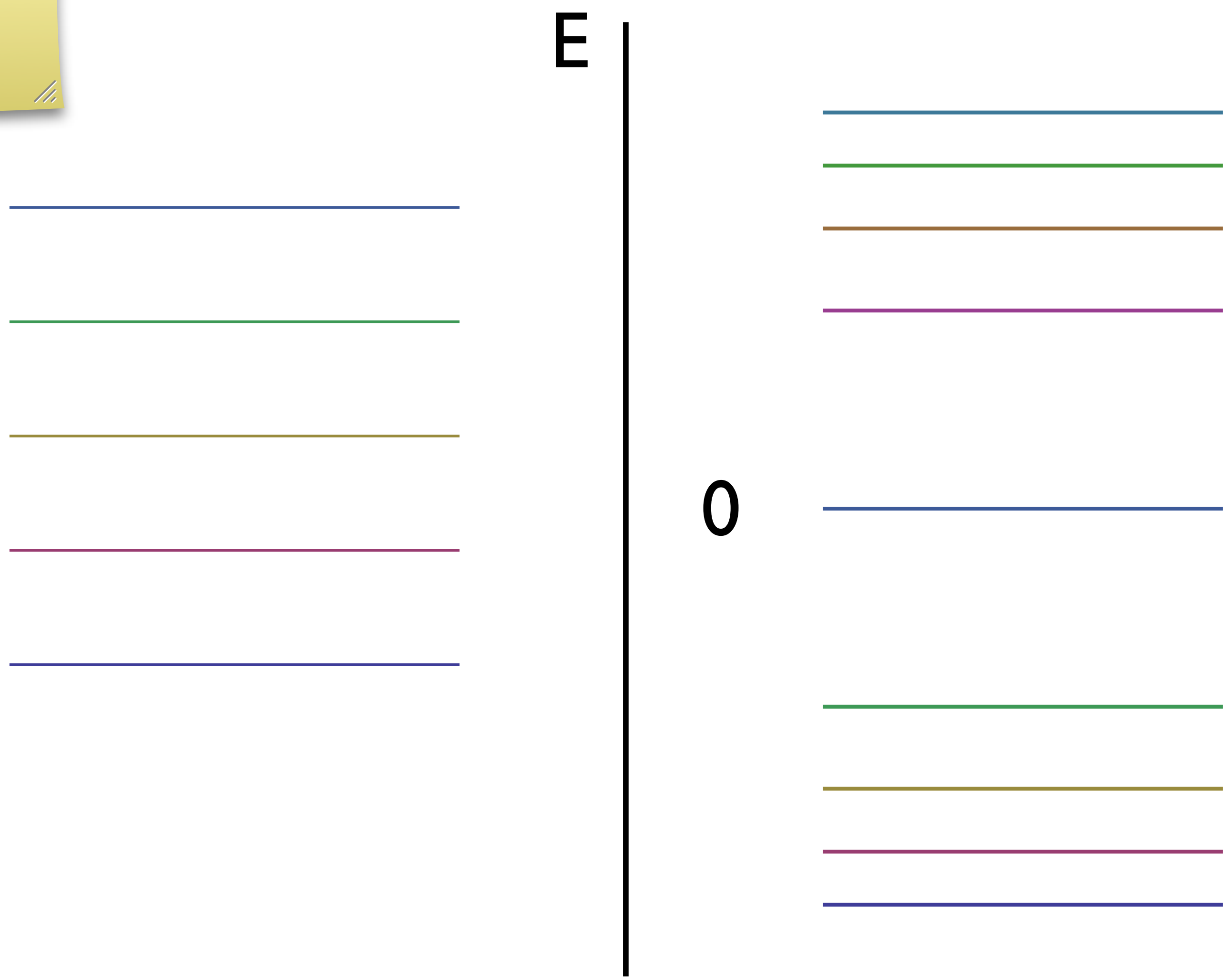
$c = \pm 1$ $\qquad$ $E = \pm\hbar\omega_c\sqrt{n}$

Relativistic vs NR LLs

A semiconductor 2DEG is formed by doping electrons into the conduction band.

Fermi in a semiconductor 2DEG is usually “high”

$\omega_c/2$

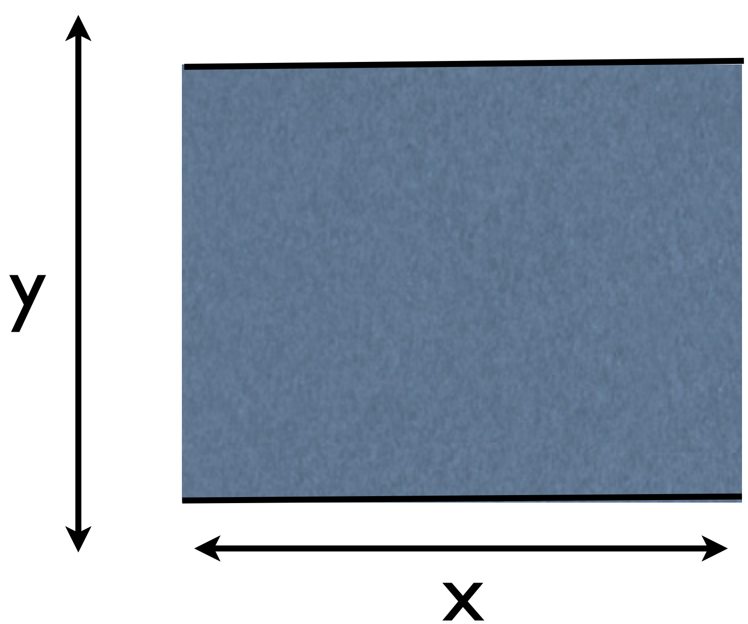


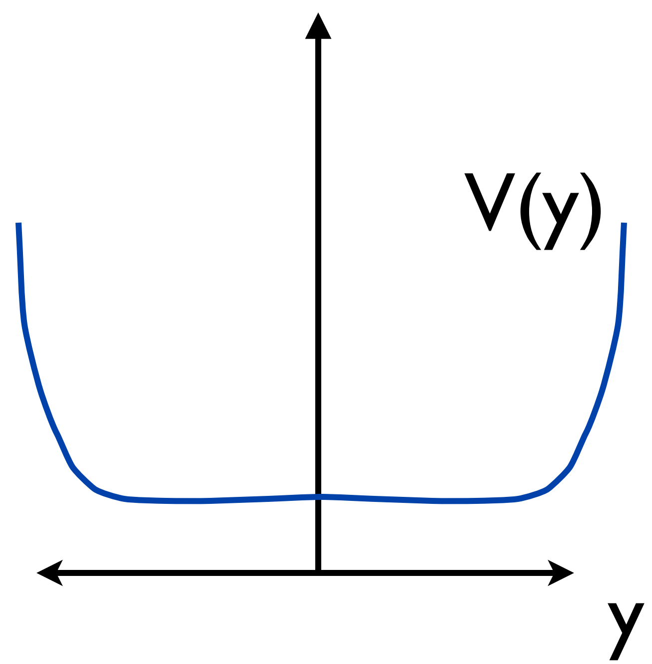
Fermi level is “in the middle” of 0th LL in undoped graphene

This is because there are a lot of electrons in graphene: 1 per C atom, filling the “negative” energy LLs

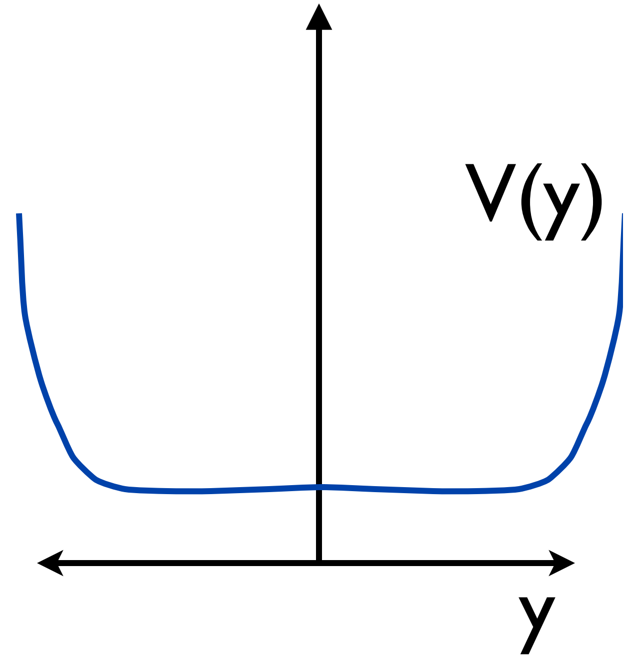
Edge states

- A simple way to understand the quantization of Hall effect, realized by Halperin
- Consider Hall bar





Edge states



$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{1}{2}m\omega_c^2\left(y-k_x\ell^2\right)^2+V(y)\right]Y=\epsilon Y$$

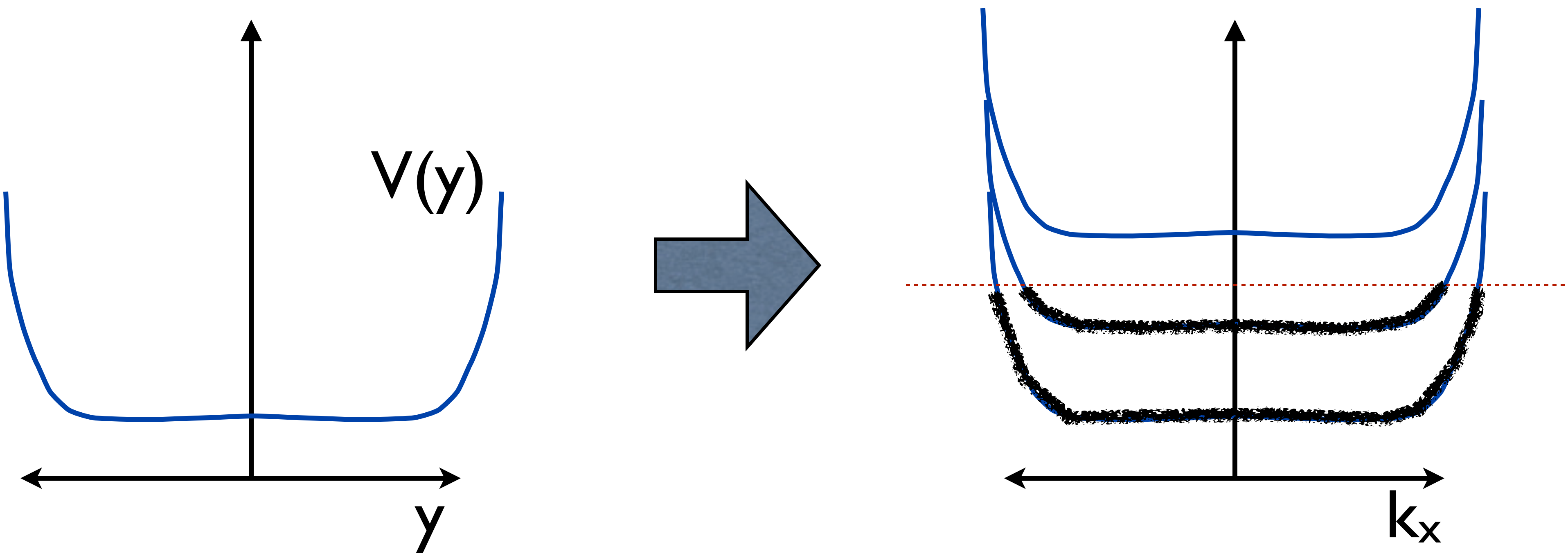
If V(y) is slowly varying, then we can approximate

$$V(y)\approx V(k_x\ell^2)$$

$$\epsilon_n\approx\hbar\omega_c(n+\tfrac{1}{2})+V(k_x\ell^2)$$

Edge states

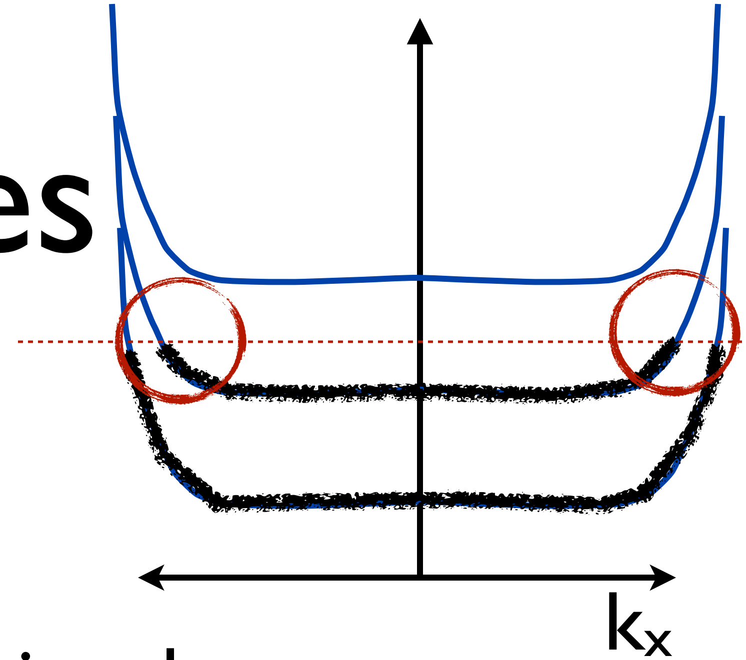
$$\epsilon_n \approx \hbar\omega_c(n + \tfrac{1}{2}) + V(k_x \ell^2)$$



Low energy states at the edges of the system

Edge states

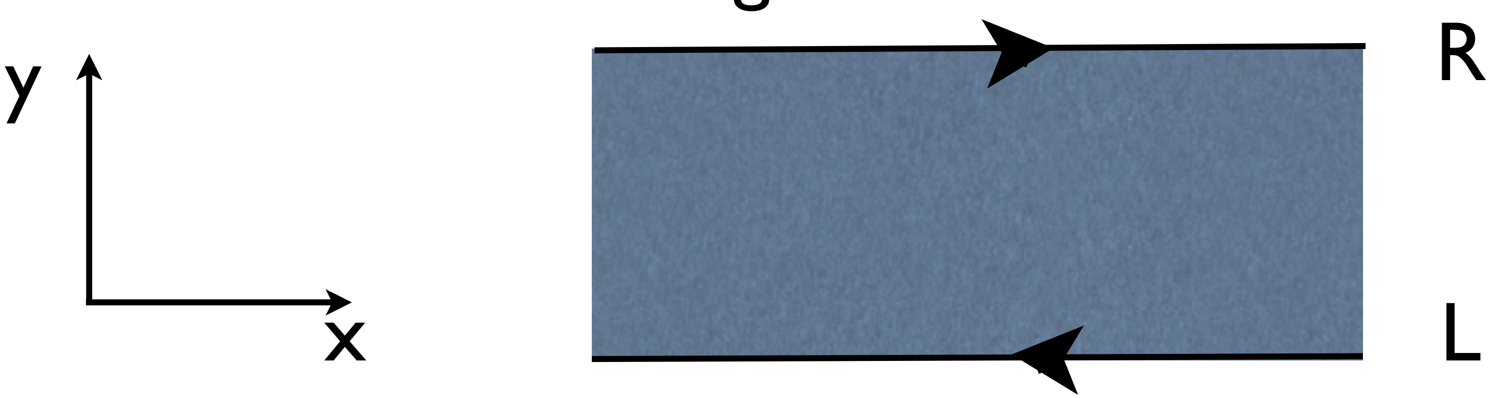
$$\epsilon_n \approx \hbar\omega_c(n+\tfrac{1}{2}) + V(k_x\ell^2)$$



- Near the edge, we can linearize the energy

$$k_x = \pm K_n + q_x \qquad\qquad \epsilon_n \approx \epsilon_F \pm v_n q_x$$

- This describes "right and left-moving chiral fermions" = edge states



Edge states

- Corresponds to semi-classical “skipping orbits”