

Physics 123B: Mid-Term
February 7, 2013

1. Consider a square lattice with lattice spacing 1, and suppose the density of electrons = $3.14159 \dots$ electrons per unit cell.

- (a) Plot the free electron Fermi surface in the reduced zone scheme, i.e. within the First Brillouin zone.

First we need to find the reciprocal lattice and first Brillouin zone. For the square lattice, with spacing 1, the reciprocal lattice is also square, with lattice spacing 2π . So there are points at $(\pm 2\pi, 0), (0, \pm 2\pi)$. Then by the usual construction, the first Brillouin zone is the square $|k_x| < \pi, |k_y| < \pi$. Now the free electron Fermi surface is of course a circle. Its radius is determined by the density of electrons. As usual, the electron density is equal to twice (for spin) the area inside the Fermi surface, divided (in two dimensions) by $(2\pi)^2$. Hence

$$2 \times \pi k_F^2 \times \frac{1}{(2\pi)^2} = n = \pi, \quad k_F = \sqrt{2}\pi. \quad (1)$$

For this radius, the free electron circle *exactly intersects the Brillouin zone corners*.

All of the Fermi surface is outside the first Brillouin zone, except the intersection points, forming four wedges. Each wedge can be translated back by a reciprocal lattice vector into the 1BZ, as shown (in the first scanned page). The result is a sort of four-pointed star (also shown).

- (b) Sketch how this surface might be expected to deform when a weak periodic potential is taken into account.

In general, we expect that once the potential is included, the Fermi surface will become smooth. Actually there are two ways we might deform the surface in part (a), either rounding the points inward, or breaking them apart. In the first case, we get a star with rounded points. In the second, we actually have two "lens"-shaped Fermi surfaces which straddle the zone boundaries. Either answer is fine. Both are shown in the first scanned page.

- (c) Is the resulting surface an electron or hole Fermi surface?

If we take the rounded star, the interior of the surface corresponds to the outside of the original Fermi surface. So this is a hole surface. If we take the two lenses, then these are electron Fermi surfaces.

2. Consider a tight-binding model with only nearest-neighbor hopping γ for the cubic lattice. Find the bandwidth, i.e. the difference between the maximum and minimum energy of the tight-binding band.

The tight binding equation is

$$\epsilon_0 \psi_R - \gamma \sum_{a=1}^3 (\psi_{R+a\hat{x}_a} + \psi_{R-a\hat{x}_a}) = \epsilon \psi_R. \quad (2)$$

We solve this by taking $\psi_R = e^{i\mathbf{k}\cdot\mathbf{R}}$. This gives

$$\epsilon(k) = \epsilon_0 - 2\gamma \sum_{a=1}^3 \cos k_a a. \quad (3)$$

The minimum energy is achieved when the cosines = +1, i.e. $\mathbf{k} = (0, 0, 0)$ So $\epsilon_{\min} = \epsilon_3 - 6\gamma$. The maximum is achieved when the cosines are -1, i.e. $\mathbf{k} = \pi/a(1, 1, 1)$, so $\epsilon_{\max} = \epsilon_0 + 6\gamma$. Thus the bandwidth is $W = \epsilon_{\max} - \epsilon_{\min} = 12\gamma$.

3. Draw the Brillouin zone for graphene, and mark the location of the Dirac points (no need to specify dimensions/lengths in your drawing). In addition, draw the Fermi surface(s) which would occur if you were to add a small density of additional electrons (e.g. a tenth of an electron per unit cell).

The Brillouin zone and Dirac points were meant to be drawn from memory. See second scanned page. The Fermi surface consists of small circles centered on the Dirac points, as explained there.

4. Give two methods of experimentally measuring the Fermi surface in a metal.

Acceptable answers include: quantum oscillations, or more specifically Shubnikov de Haas oscillations, or de Haas van Alphen oscillations, angle resolved photoemission, STM tunneling interference oscillations, positron annihilation spectroscopy...

5. For a two dimensional electron gas of large area A , how many states are in each Landau level?

There are $N = \frac{A}{2\pi\ell^2}$ states, where $\ell = \sqrt{\hbar/eB}$ is the magnetic length. Equivalently, the number of states is equal to the number of flux quanta piercing the system, $N = BA/\varphi$, with $\varphi = h/e$ is the flux quantum.

6. What are the values of the Hall conductivity σ_{xy} and diagonal conductivity σ_{xx} in the integer quantum Hall state, in terms of fundamental constants?

On a Hall plateau, $\sigma_{xy} = n\frac{e^2}{h}$, and $\sigma_{xx} = 0$.

7. Explain why the quantum Hall effect is robust.

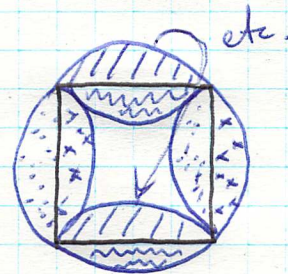
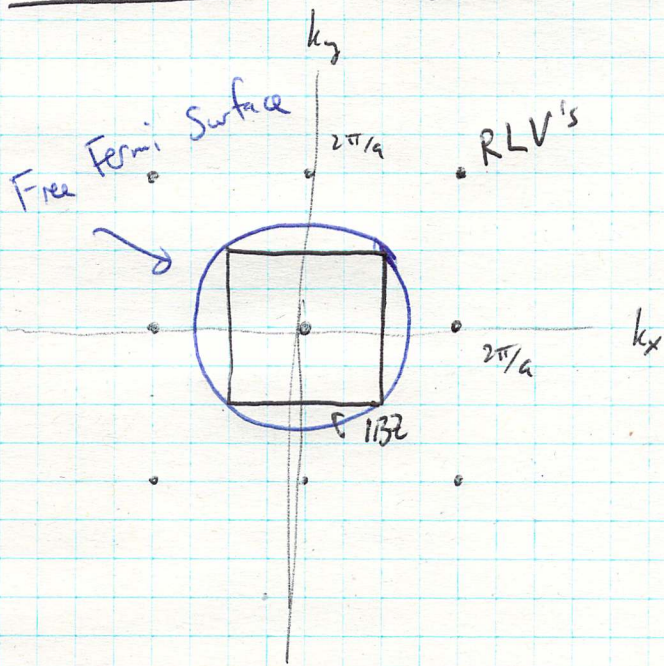
The quantum Hall effect is robust because it exists so long as there are edge states at opposite sides of the sample, which carry current in one direction only and are in separate equilibrium. The states on a single edge are *chiral*, that is, they propagate only in one direction. Therefore even if an electron scatters at the edge, this does not degrade the current. Because the edge states are separated by a macroscopic distance, between which are only localized levels, it is extremely difficult for an electron to scatter from one edge to the other.

8. In the quantum spin Hall effect, what is the dispersion relation (i.e. energy versus wavevector) of the edge states with up and down spin at one side of the sample? Based on this, if a Hall voltage is applied between the top and bottom sides of the sample (i.e. in the y direction), is there a net electrical current flow in the normal (i.e. the x) direction?

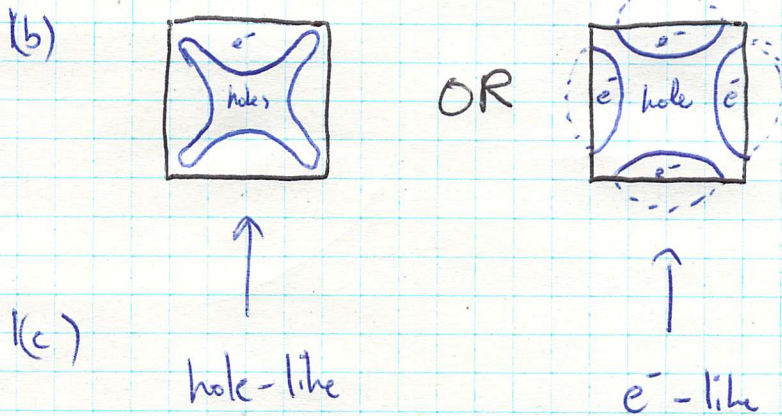
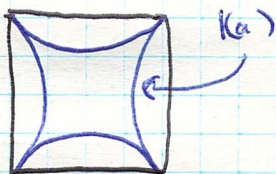
The exact answer here depends upon the details. However, edge states with up and down spin propagate in *opposite* directions. So for example, we may take $\epsilon(q) = \epsilon_F + \sigma vq$, where $\sigma = +1$ for up spins and -1 for down spins, v is the edge velocity, and $q = k - K$, where k is the momentum along the edge, and K is the momentum at which the edge state crosses the Fermi energy.

If a Hall voltage is applied, *both* edge states at a single edge are shifted in their chemical potential. This induces an equal number of additional electrons propagating in both directions, and hence no electrical current flows. In fact a spin current flows.

Problem 1 Figures

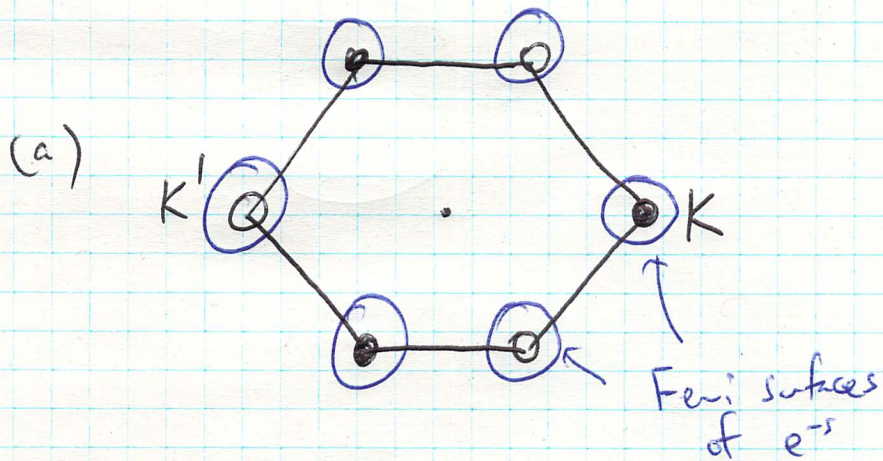


like-shaded regions are equivalent by shifts of RLVs.



Problem 3

Brillouin zone is a hexagon

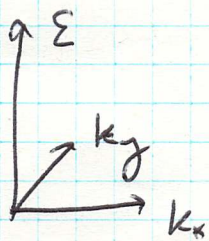
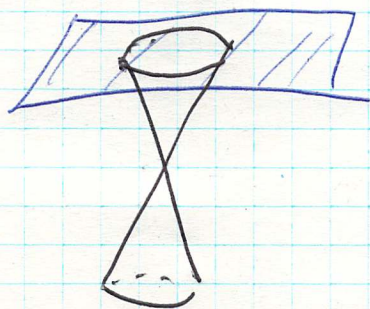


● points are equivalent
○ " " " "

(b) ~~When~~ If we add e^- , they will go into the lowest energy unoccupied states, i.e. the Fermi energy will increase. The low energy states are those near the Dirac points.

$$\epsilon_+ = v|k - K| \quad \text{or} \quad v|k - K'|$$

Setting $\epsilon_+ = \epsilon_F$ (ϵ_- states are full), we get small Fermi circles



you can also think of the circles as intersections of the Dirac cones with $\epsilon = \epsilon_F$ planes.