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Diffraction Grating Equation with Example Problems[1]

1 Grating Equation

In Figure 1, parallel rays of monochromatic radiation, from a single beam in the form of rays 1 and 2, are incident on a (blazed) diffraction grating at an angle θ_i relative to the grating normal. These rays are then diffracted at an angle $-\theta_r$. We use a negative sign here based on our convention of angle definition. That is, positive angles are measured to the left of the grating normal, negative angles to the right. From this figure we can see that ray 2 travels a greater distance than ray 1. This overall difference can be computed by subtracting the increase in path lengths from rays 1 and 2.

We can analyze this in more detail by examining Figure 2. Here we represent the change in path lengths for beams 1 and 2 as

$$\Gamma_i = d\sin\theta_i \tag{1}$$

$$\Gamma_r = d\sin(-\theta_r) = -d\sin\theta_r \tag{2}$$

Thus the total difference in path length is

$$\Gamma_T = d\sin\theta_i - (-d\sin\theta_r) = d(\sin\theta_i + \sin\theta_r) \tag{3}$$

For constructive interference to occur this difference must equal and integer multiple of the wavelength, λ. That is

$$\Gamma_T = m\lambda \tag{4}$$

[1]Submit suggestions and typos to emmett@cis.rit.edu

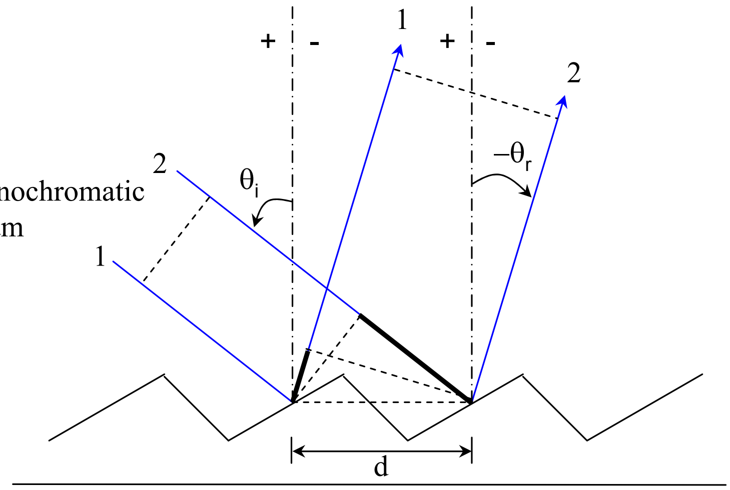

Figure 1: Monochromatic beam incident on (blazed) diffraction grating at angle θ_i and diffracted at angle $-\theta_r$. The blaze spacing is d.

where $m = \pm 0, 1, \ldots, k$ and k is a smaller integer. m is called the diffraction *order*. Upon substitution we have

$$\boxed{m\lambda = d(\sin\theta_i + \sin\theta_r)} \tag{5}$$

which is the general grating equation using the positive/negative angle conventions stated earlier.

In the event that one treats all angles as positive, the grating equation is sometime defined as

$$m\lambda = d(\sin\theta_i \pm \sin\theta_r) \tag{6}$$

where one uses the "subtraction" operation when θ_i and θ_r are *opposite* sides of the grating normal. This is the same as using Eq. (5) with a negative diffraction angle. The same convention holds for *transmission* gratings as well.

1.1 Location of the $m = 0$ Order

The location of the $m = 0$ order will always be found at an angle equal and opposite to the incident angle. This can be shown by setting $m = 0$ in Eq. (5)

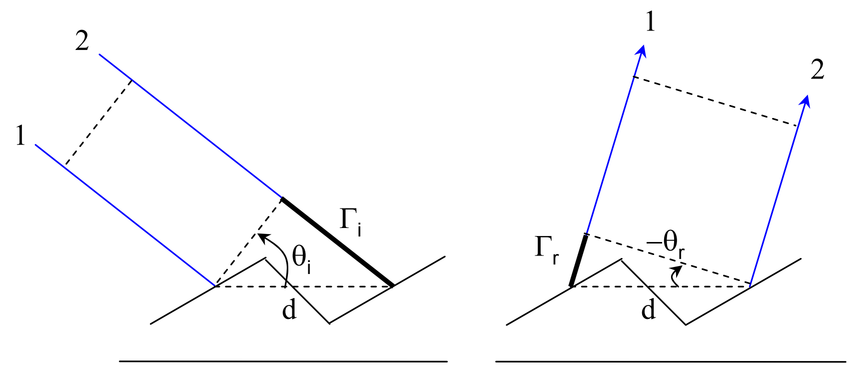

Figure 2: Illustration of path difference between incident and diffracted rays.

such that

$$\begin{aligned} 0 &= d(\sin\theta_i + \sin\theta_r) \qquad (7) \\ -d\sin\theta_r &= d\sin\theta_i \\ -\theta_r &= \theta_i \end{aligned}$$

1.2 When the Incident Angle, $\theta_i = 0$

Often, the incident illumination is perpendicular to the gratings surface. That is, the incident angle is zero. In this case the grating formulation of Eq. (5) reduces to

$$m\lambda = d\sin\theta_r \qquad (8)$$

which is the diffraction grating equation for *normal* incidence.

2 Example Problems

Problem 1. A grating has 8000 slits ruled across a width of 4 cm. What is the wavelength, and the color, of the light whose two fifth-order maxima subtend an angle of 90 degrees?

Solution:

This problem is illustrated in Figure 3. The general form of the grating equation is

$$m\lambda = d(\sin\theta_i + \sin\theta_r)$$

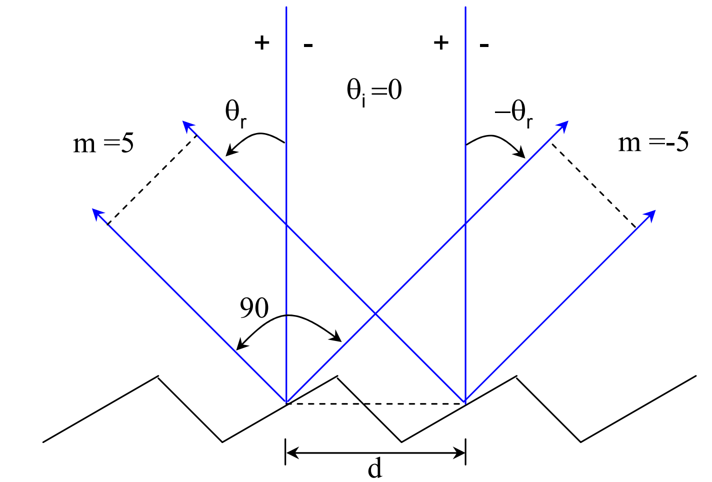

Figure 3: Grating geometry for problem 1.

where m is the diffraction order, λ is the wavelength, d is the groove (slit) spacing, θ_i is the incident angle, and θ_r is the diffracted angle. Here the incident angle $\theta_i = 0$. Note that 90 degrees is the angle subtended by two maxima, one on each side. Thus $\theta_r = 45°$ for $m = +5$ and $-\theta_r = 45°$ for $m = -5$. Therefore, for the order $|m = 5|$ we have

$$\lambda = \frac{d(\sin\theta_r)}{m} = \frac{\left(\frac{0.04\,m}{8000}\right)\sin 45}{5} = \boxed{707.1\,nm}$$

Since 650 nm is roughly red, 707 nm probably corresponds to a color of dark red.

Problem 2. A diffraction grating 3 cm wide produces a deviation of 33 degrees in the second order with light of wavelength 600 nm. What is the total number of lines on the grating?

Solution:

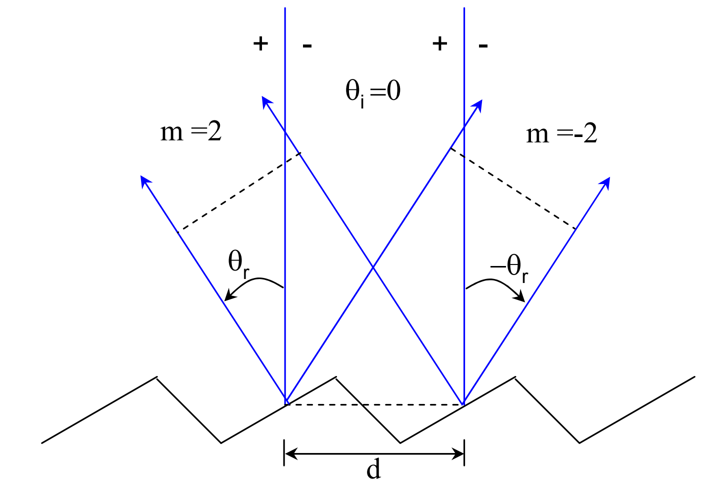

Figure 4: Grating geometry for problem 2.

The incident angle is zero. Therefore the slit spacing is

$$d = \frac{m\lambda}{\sin\theta_r} = \frac{(2)(600\,nm)}{\sin 33} = 2.2 \times 10^{-6}\,m$$

$$N = \frac{\ell}{d} = \frac{0.03\,m}{2.2 \times 10^{-6}\,m} = \boxed{13,616\ lines}$$

Problem 3. A diffraction grating has 12,600 rulings uniformly spaced over 25.4 mm. It is illuminated at normal incidence by yellow light from a sodium vapor lamp. This light contains two closely spaced lines (the well-known sodium doublet) of wavelengths 589 nm and 589.59 nm. At what angle will the first order maximum occur for the first of these wavelengths?

Solution:

The incident angle, θ_i is zero. The grating spacing d is given by

$$d = \frac{\ell}{N} = \frac{25.4 \times 10^{-3}\,m}{12600} = 2.016 \times 10^{-6}\,m$$

The first-order maximum corresponds to $m = 1$ and $m = -1$. We thus have

$$\theta_{r\_589} = \sin^{-1}\left(\frac{m\lambda}{d}\right) = \sin^{-1}\left(\frac{(1)(589 \times 10^{-9}\,m)}{2.016 \times 10^{-6}\,m}\right) = \boxed{\pm 17\,deg}$$

Problem 4. An echellette grating containing 1450 blazes (lines) per millimeter was irradiated with a polychromatic beam at an incident angle of 48 degrees to the grating normal. Calculate the wavelengths of radiation that would appear at an angle of reflection of +20, +10, 0, and -10 degrees for the first order, as can be seen in Figure 5. Recompute for the second and third orders as well.

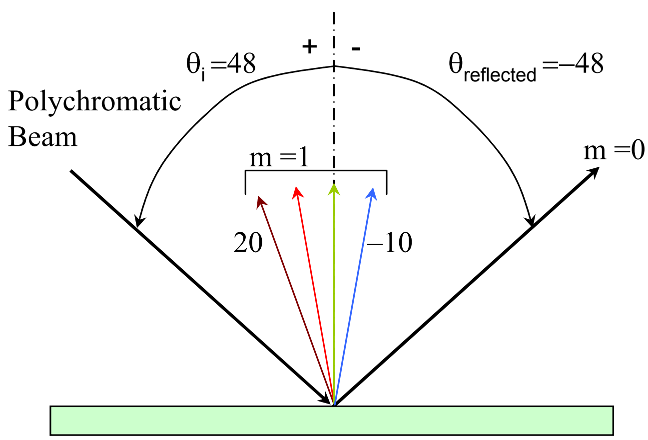

Figure 5: Grating geometry for problem 4.

Solution:

To obtain d we write

$$d = \frac{\ell}{N} = \frac{0.001\,m}{1450} = 6.90 \times 10^{-7}\,m$$

The incident angle, $\theta_i = +48$ degrees. When the diffracted angle, θ_r equals +20 degrees and $m = 1$ we have

$$\lambda = \frac{(6.90 \times 10^{-7}\,m)\Big(\sin(+48) + \sin(+20)\Big)}{(1)} = \boxed{749\,nm}$$

The wavelengths for the second and third orders are 374 and 250 nm, respectively. Similarly, when the diffracted angle angle is $\theta_r = -10$ degrees and $m = 1$ we have

$$\lambda = \frac{(6.90 \times 10^{-7}\, m)\Big(\sin(+48) + \sin(-10)\Big)}{(1)} = \boxed{393\, nm}$$

Further calculations of a similar kind yield the following data:

Diffraction Angle (Degrees)	m=1 (nm)	m=2 (nm)	m=3 (nm)
20	749	374	249
10	632	316	211
0	513	256	171
-10	393	196	131

Problem 5. You have been handed a transmission grating by your supervisor who wants to know how widely the red light and blue light fringes, in second order, are separated on a screen one meter from the grating. You are told that the separation distance between the red and blue colors is a critical piece of information needed for an experiment with a grating spectrometer. The transmission grating is to be illuminated at normal incidence with red light at $\lambda = 632.8\, nm$ and blue light at $\lambda = 420.0\, nm$. Printed on the frame surrounding the ruled grating, you see that there are 5000 slits (lines) per centimeter on this grating.

Solution:

Since there are 5000 slits or grooves per centimeter, you know that the distance, d between the slits, center to center, must be

$$d = \frac{\ell}{N} = \frac{1 \times 10^{-2}\, m}{5000} = 2 \times 10^{-6}\, m$$

The incident angle is zero. So for $m = +2$, for example, we have

$$\theta_{r(red)} = \sin^{-1}\left(\frac{m\lambda}{d}\right) = \sin^{-1}\left(\frac{(2)(632.8 \times 10^{-9}\, m)}{2 \times 10^{-6}\, m}\right) = 39.3°$$

$$\theta_{r(blue)} = \sin^{-1}\left(\frac{m\lambda}{d}\right) = \sin^{-1}\left(\frac{(2)(420.0 \times 10^{-9}\, m)}{2 \times 10^{-6}\, m}\right) = 24.8°$$

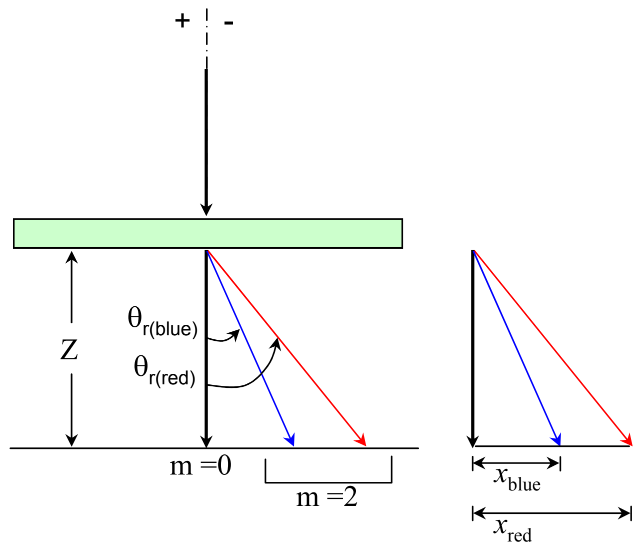

Figure 6: Grating geometry for problem 5.

From geometry, we can calculate each distance (x_{red} and x_{blue}) on the screen, as measured from the $m = 0$ order as

$$\begin{aligned} x_{red} &= Z \tan \theta_{r(red)} \\ x_{blue} &= Z \tan \theta_{r(blue)} \end{aligned}$$

where Z is the distance to the screen. The difference between these distances, Δx is the separation we are looking for. That is

$$\begin{aligned} \Delta x &= x_{red} - x_{blue} \\ &= Z \tan \theta_{r(red)} - Z \tan \theta_{r(blue)} \\ &= Z(\tan \theta_{r(red)} - \tan \theta_{r(blue)}) \\ &= (1\, m)(\tan 39.3 - \tan 24.8) \\ &= \boxed{35.6\,\mathrm{cm}} \end{aligned}$$