Elementary Statistical Physics

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C. KITTEL

Professor of Physics
University of California, Berkeley, California

29. The Nyquist Theorem

Reference: H. Nyquist, *Phys. Rev.* **32,** 110 (1928).

The Nyquist theorem is of great importance in experimental physics and in electronics. The theorem gives a quantitative expression for the thermal noise generated by a system in thermal equilibrium and is therefore needed in any estimate of the limiting signal-to-noise ratio of an experimental apparatus. In the original form the Nyquist theorem states that the mean square voltage across a resistor of resistance R in thermal equilibrium at temperature T is given by

$$\overline{V^2} = 4RkT\,\Delta f, \tag{29.1}$$

where Δf is the frequency band width within which the voltage fluctuations are measured; all Fourier components outside the given range are ignored. Recalling from Sec. 28 the definition of the spectral density $G(f)$, we may write the Nyquist result as

$$G(f) = 4RkT. \tag{29.2}$$

This is not strictly the power density, which would be $G(f)/R$. We do not write τ for kT in this section, to avoid confusion with the correlation or relaxation time. The maximum thermal noise power per unit frequency range delivered by a resistor to a matched load will be $G(f)/4R = kT$; the factor of 4 enters where it does because the power delivered to the load R' is

$$\overline{I^2}R' = \overline{V^2}R'/(R + R')^2,$$

which at match ($R' = R$) is $\overline{V^2}/4R$. The circuit is shown in Fig. 29.1.

In this section we derive the Nyquist theorem in two ways: first, following the original transmission line derivation, and, second, using a microscopic argument. Still another derivation is indicated in Sec. 30.

Transmission Line Derivation

Consider as in Fig. 29.2 a lossless transmission line of length l and characteristic impedance $Z_c = R$ terminated at each end by a resistance R. The line is therefore matched at each end, in the sense that all energy traveling down the line will be absorbed without reflection

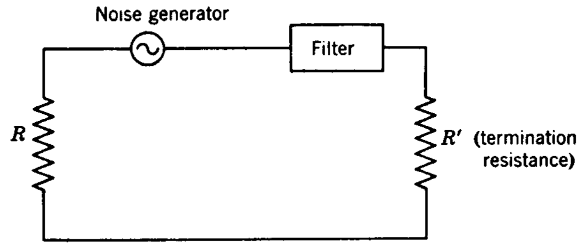

Fig. 29.1. The noise generator produces a power spectrum $G(f) = 4RkT$. If the filter passes unit frequency range, the resistance R' will absorb power $2RkT$. R' is matched to R.

in the appropriate resistance. The entire circuit is maintained at temperature T.

In analogy to the argument in Sec. 22 on black-body radiation, the transmission line has two electromagnetic modes (one propagating in each direction) in the frequency range

$$\delta f = \frac{c'}{l}, \tag{29.3}$$

where c' is the propagation velocity on the line. Each mode has energy

$$\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

in equilibrium, according to (22.2). We are usually concerned here with the classical limit $\hbar\omega \ll kT$, so that the thermal energy per mode may be taken as kT. Thus the energy on the line in the frequency

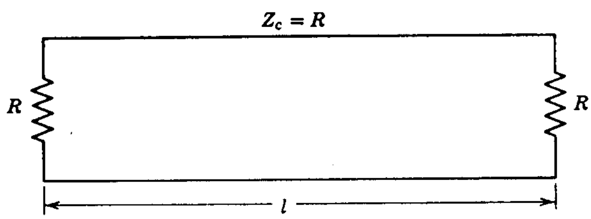

Fig. 29.2. Transmission line of length l with matched terminations, as conceived for the derivation of the Nyquist theorem.

range Δf is

$$2kTl\,\Delta f/c'. \tag{29.4}$$

The rate at which energy comes off the line in *one* direction is

$$kT\,\Delta f. \tag{29.5}$$

The power coming off the line at one end is all absorbed in the terminal impedance R at that end; there are no reflections because the terminal impedance is matched to the line. The load emits energy at the same rate. The power input to the load is

$$\overline{I^2}R = kT\,\Delta f,$$

but $V = I(2R)$, so that

$$\overline{V^2}/R = 4kT\,\Delta f, \tag{29.6}$$

which is the Nyquist theorem.

Microscopic Derivation

We consider a resistor of resistance R with N electrons per unit volume; length l; area A; and carrier relaxation time τ_c. We treat the electrons as Maxwellian, but it will be shown later that the noise voltage is independent of such details, involving only the value of the resistance regardless of the details of the mechanisms contributing to the resistance.

First note that

$$V = IR = RAj = RANe\bar{u}; \tag{29.7}$$

here V is the voltage, I the current, j the current density, and $\bar{u}$ is the average (or drift) velocity component of the electrons down the resistor. Observing that NAl is the total number of electrons in the specimen,

$$NAl\bar{u} = \Sigma\, u_i, \tag{29.8}$$

summed over all electrons. Thus

$$V = (Re/l)\,\Sigma\, u_i = \Sigma\, V_i, \tag{29.9}$$

where

$$V_i = Reu_i/l. \tag{29.10}$$

Now u_i is a random variable; V_i is also a random variable. The spectral density $G(f)$ has the property that in the range Δf

$$\overline{V_i^2} = G(f)\,\Delta f. \tag{29.11}$$

We suppose that the correlation function may be written as

$$C(\tau) = \overline{V_i(t)\,V_i(t+\tau)} = \overline{V_i^2}e^{-\tau/\tau_c}, \tag{29.12}$$

where τ_c is the relaxation time or mean time of flight of the conduction electrons; this assumption stands up under detailed examination. Then, from the Wiener-Khintchine theorem we have

$$\begin{aligned} G(f) &= 4(Re/l)^2\overline{u^2}\int_0^\infty e^{-\tau/\tau_c}\cos 2\pi f\tau\,d\tau \\ &= 4(Re/l)^2\overline{u^2}\,\frac{\tau_c}{1+(2\pi f\tau_c)^2}, \end{aligned} \tag{29.13}$$

using our previous results. Usually in metals at room temperature $\tau_c < 10^{-13}$ sec, so from dc through the microwave range $2\pi f\tau_c \ll 1$ and may be neglected. We recall that

$$\tfrac{1}{2}m\overline{u^2} = \tfrac{1}{2}kT, \tag{29.14}$$

so that

$$\overline{u^2} = kT/m. \tag{29.15}$$

Thus in the frequency range Δf

$$\begin{aligned} \overline{V^2} &= NAl\overline{V_i^2} = NAl\,G(f)\,\Delta f \\ &= NAl4\left(\frac{kT}{m}\right)\left(\frac{Re}{l}\right)^2\tau_c\,\Delta f, \end{aligned} \tag{29.16}$$

or

$$\boxed{\overline{V^2} = 4kTR\,\Delta f,} \tag{29.17}$$

the desired result. Here we have used the relation

$$\sigma = Ne^2\tau_c/m \tag{29.18}$$

from the theory of conductivity and also the elementary relation

$$R = l/\sigma A; \tag{29.19}$$

σ is the electrical conductivity.

The simplest way to establish (29.18) in a plausible way is to solve the drift velocity equation

$$m\left(\frac{d}{dt} + \frac{1}{\tau_c}\right)\bar{u} = eE, \tag{29.20}$$

so that in the steady state (or for $\omega\tau_c \ll 1$) we have

$$\bar{u} = e\tau_c E/m, \tag{29.21}$$

giving for the mobility (drift velocity per unit electric field)

$$\mu = \bar{u}/E = e\tau_c/m. \tag{29.22}$$

Then we have for the electrical conductivity

$$\sigma = j/E = Ne\bar{u}/E = Ne^2\tau_c/m. \tag{29.23}$$

We now prove that our assumption of a classical (Maxwellian) distribution of electron velocities can have no effect on the noise power. Let us take as in Fig. 29.3 a Fermi-Dirac wire of resistance R connected by a transmission line of characteristic impedance $Z_c = R$ to a Maxwellian wire of resistance R. Because $Z_c = R$ the transmission line is matched to both resistors and delivers to each one all the thermal noise power transmitted by the other resistor. If both resistors are initially at the same temperature T, one wire must not heat up at the expense of the other wire. Therefore the noise powers produced are equal, which proves that $\overline{V^2}$ depends only on R and not on the details of the conductivity mechanism. We really ought to establish that the power spectra are equal at any frequency: this may be accomplished by putting a narrow band-pass filter in the circuit. The above argument may now be applied to the power at the frequencies passed by the filter.

The dependence of $\overline{V^2}$ on R and on T was first investigated carefully by J. B. Johnson, *Phys. Rev.* **32,** 97 (1928). Johnson determined the Boltzmann constant k from the noise power and obtained a value

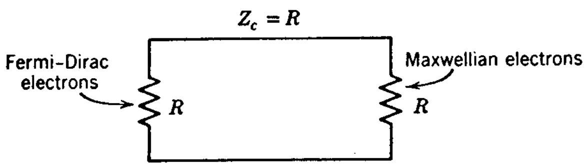

Fig. 29.3. Arrangement to illustrate that the Nyquist theorem is independent of the electron distribution. The power emitted by one resistor is absorbed in the other. At thermal equilibrium the emitted powers must be equal, or else one resistor would heat up at the expense of the other.

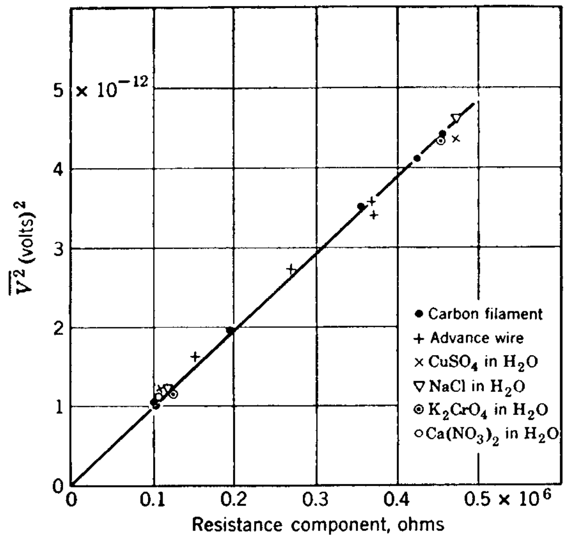

Fig. 29.4. Voltage squared versus resistance for various kinds of conductors. (After J. B. Johnson.)

within 8 per cent of the correct value. His results exhibiting the dependence of $\overline{V^2}$ on R at constant temperature and Δf are shown in Fig. 29.4.

Exercise 29.1. Express $\overline{V^2}/R$ in watts for a 5000-cycle band width at 300°K.

Exercise 29.2. Prove, with reference to (29.12), that a one-dimensional Gaussian process will be Markoffian only for a correlation function $C(\tau) \propto e^{-\tau/\tau_0}$. This result is due to Doob. To show the theorem calculate the distribution functions $p_3(y_1y_2y_3)$ and $p_2(y_1y_2)$, using the method of Rice. This lets us determine the conditional probability $P_3(y_1y_2|y_3)$, which for a Markoff process must be identical with $P_2(y_2|y_3)$. The equality can only be satisfied if

$$C(t_3 - t_1) = C(t_2 - t_1)\, C(t_3 - t_2),$$

which has the solution $C(\tau) \propto e^{-\tau/\tau_0}$.

Exercise 29.3. Show that the power spectrum of the random voltage associated with an impedance

$$Z(f) = R(f) + i\, Y(f)$$

is

$$G(f) = 4\, R(f)kT,$$

defining $G(f)$ as in (29.2).