Commonly Encountered Probability Distributions

Physics 129L   Fall 2021

Binomial Processes

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- The standard deviation $\sigma = \sqrt{Npq}$. **Important!**

Binomial Processes

Example: toss a coin 100 times, win if heads, lose if tails:

$$N = 100$$

$$p = 0.5$$

$$q = 1 - p = 0.5$$

$$\mu = Np = 50 \text{ heads}$$

$$\sigma = \sqrt{Npq} = \sqrt{25} = 5$$

Gaussian (Normal) Distribution

When μ is at least a few standard deviations from both 0 and N, the binomial distribution is symmetric, and the *Gaussian distribution*

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

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A Gaussian distribution has about 68% of its probability within 1σ of μ, about 95% within 2σ, and 99.7% (all but one part in 370) within 3σ.

Gaussian (Normal) Distribution

Beware! The Gaussian approximation to the binomial distribution cannot be trusted in the tails.

To get accurate estimates far from μ, you must add the discrete binomial probabilities rather than integrating a Gaussian.

Binomial Processes

Example: a dice game — win \$5 if you throw a 6, lose \$1 otherwise

$N = 300$

$p = 1/6$

$q = 1 - p = 5/6$

$\mu = Np = 50 \text{ sixes}$

$\sigma = \sqrt{Npq} = 6.45$

Expected wins: $w = 50 \pm 6.5$

Binomial Processes

Example: loaded die? You play the dice game with your cousin:

$N = 100$

$w = 31$

Was the game fair?

$p = 0.1667$

$\mu = Np = 16.67$

$\sigma = \sqrt{Npq} = 3.73$

$w - \mu = 14.33 = 3.8\sigma! \quad p(w \geq 31) = 1/3381$

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Binomial Processes

Best estimate for cousin's die:

$$N = 100$$
$$w = 31$$

$$p_a = 31/100 = 0.31 \text{ (approximate)}$$
$$\sigma_a = \sqrt{Np_a q_a} = 4.6$$
$$p_a = 0.31 \pm 0.046.$$

Binomial Processes

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$$\sigma_a = \sqrt{N p_a q_a} = \sqrt{100 \times 0.6 \times 0.4} = 4.9$$

Margin of error from sampling is about 4.9%.

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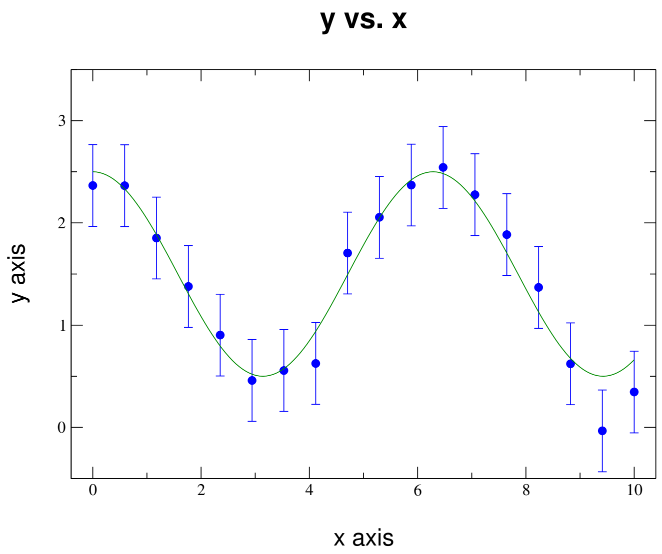
$N_a = 100/f$,

$\sigma_{\rm ct} \approx \sqrt{N_c f(1-f)}$.

Binomial Processes

What's wrong with this picture?



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$$P(\mu, k) = \frac{\mu^k}{k!} e^{-\mu}.$$

$\sigma_P = \sqrt{\mu}$ differs from the binomial σ by a factor of $\sqrt{q}$.

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- Attendance checkpoints

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  For $\mu = 1$, $P(1, 0) = e^{-1} = 0.368$.

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  For $\mu = 1$, $P(1, 2) = \frac{1}{2!}e^{-1} = 0.184.$

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- Attendance checkpoints
- Bin fluctuations