

# **Finite Difference Method: Laplace's equation**

Physics 129L Fall 2021

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This is *Laplace's equation*.

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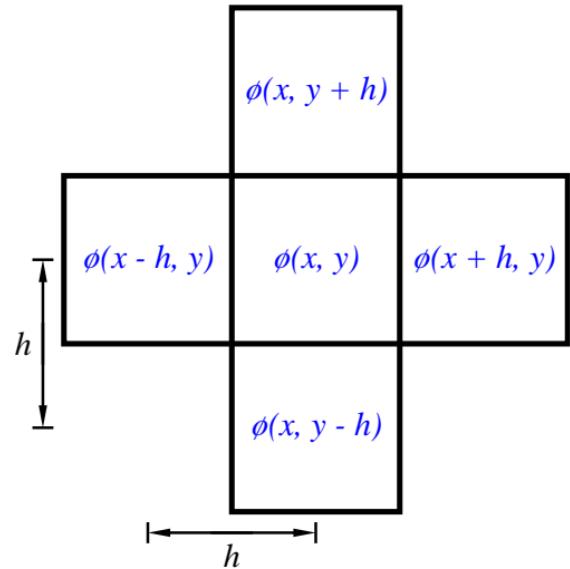
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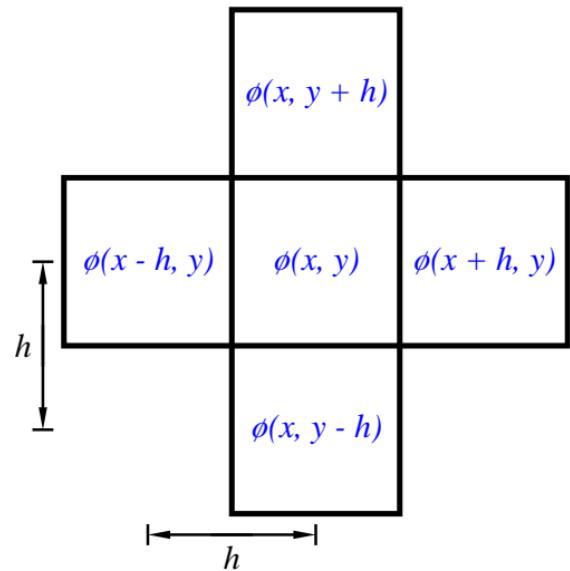


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$$\phi(x, y) = \frac{\phi(x + h, y) + \phi(x - h, y) + \phi(x, y + h) + \phi(x, y - h)}{4}$$