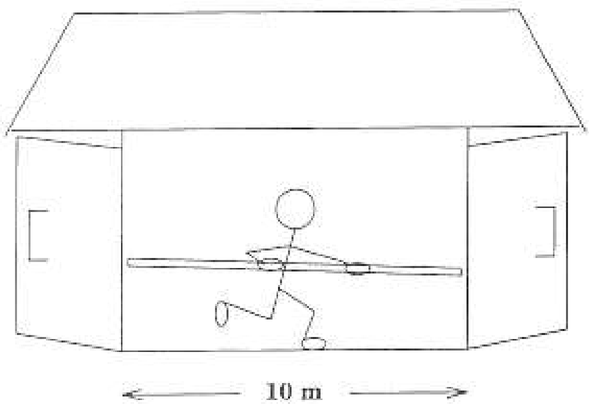
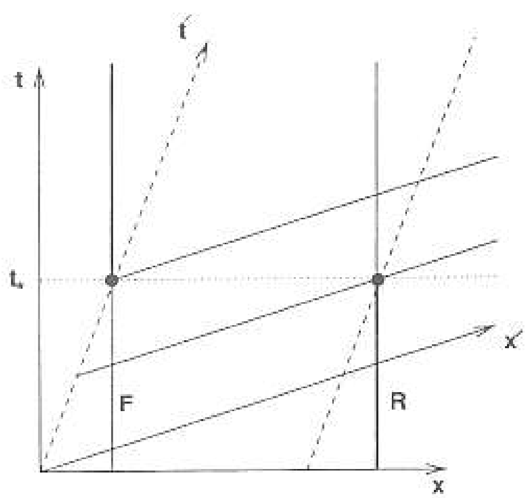
the barn. Explain, quantitatively and by means of spacetime diagrams, the apparent paradox.



Solution:



Shown above is a spacetime diagram in the frame where the barn is at rest and the pole is moving. The solid, vertical lines are the world lines of the front and rear barn doors, the heavy parts indicating when the doors are shut. The dotted lines are the world lines of the end of the pole. At the moment t_* the pole is in the barn and both doors are simultaneously shut. The coordinates (t', x') of the frame in which the pole is stationary and the barn is moving are also indicated as well as some lines of constant t'.

From this spacetime diagram it is evident that the closing of the front door and the opening of the rear door are not simultaneous in the pole frame. Rather, the front door closes after the rear door opens. This allows the shorter barn time to pass over the longer pole as we now demonstrate quantitatively.

In the pole frame the time difference between the two events simultaneous in the barn frame in [*cf.* (4.24)] is

$$\Delta t' = \gamma v L_* / c^2$$

where $L_* = 10$ m is the proper width of the barn and v is its velocity. This velocity is such that the 20 m pole is contracted to 10 m in the barn frame, i.e., $\gamma = 2$, $v = (\sqrt{3}/2)c$. At the time the rear door opens, 5 m of the pole is within the contracted 5 m length of the barn. Another $v\Delta t' = 2 \cdot (\sqrt{3}/2)^2 \cdot 10 = 15$ m can pass through before the front door closes. The total makes the full 20 m length of the pole. There is no contradiction.

4-4. *A satellite orbits the Earth in a circular orbit above the equator a distance of 200 km from the surface. By how many seconds per day will a clock on such a satellite run slow compared to a clock on the Earth? (Compute just the special relativistic effects.)*

Solution: Neglecting the earth's orbital motion, we can think of the earth as rotating about an axis in an inertial frame. The speed V_s of the satellite is related to the distance r_s from the earth's center by

$$\frac{V_s^2}{r_s} = \frac{GM_\oplus}{r_s^2}$$

where $M_\oplus$ is the mass of the earth. Thus,

$$\frac{V_s}{c} = \left(\frac{GM_\oplus}{c^2 r_s}\right)^{\frac{1}{2}} = 2.6 \times 10^{-5}$$

for $r_s = r_\oplus + 200km$, where $r_\oplus = 6378km$. The speed of the earth at its surface is

$$\frac{V_{\rm surf}}{c} = \frac{1}{c}\left(\frac{2\pi r_\oplus}{24hrs}\right) \doteq 1.5 \times 10^{-6}$$

The satellite clock is moving faster in the inertial frame and will run slower compared with the clock on the surface. The ratio of rates is

$$\frac{(\text{Rate of sat clock})}{(\text{Rate of surf clock})} = \frac{(1 - V_s^2/c^2)^{\frac{1}{2}}}{(1 - V_{\text{surf}}^2/c^2)^{\frac{1}{2}}} \approx 1 + \frac{1}{2}\left(\frac{V_{\text{surf}}}{c}\right)^2 - \frac{1}{2}\left(\frac{V_s}{c}\right)^2$$

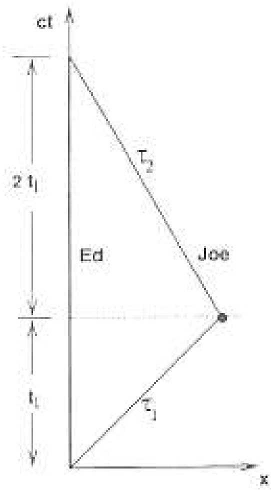
since both velocities are small compared to c. The ratio is thus

$$1 + 3.4 \times 10^{-10}$$

So, in one day the clocks will differ by $(3.4 \times 10^{-10}) \times (8.6 \times 10^4 \text{ s}) = 29\ \mu\text{s}$.

4-9. *Consider two twins Joe and Ed. Joe goes off in a straight line traveling at a speed of (24/25)c for seven years as measured on his clock, then reverses and returns at half the speed. Ed remains at home. Make a spacetime diagram showing the motion of Joe and Ed from Ed's point of view. When they return what is the difference in ages between Joe and Ed?*

Solution:



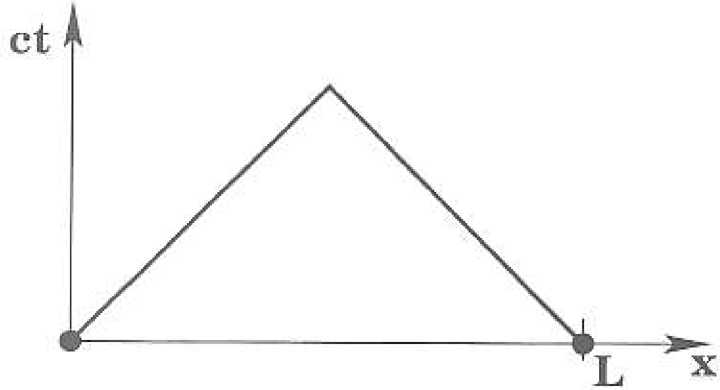
The time t_1 to the turn around point as measured by Ed is related to Joe's proper time, $\tau_1 = 7$ yr to the same point by

$$t_1 = \tau_1 \left[1 - \left(\frac{24}{25}\right)^2\right]^{-\frac{1}{2}} = 25 \text{ yr}$$

Since the return velocity is half the outbound velocity, it takes twice as long for the return trip (50 yr) according to Ed. Ed has therefore aged by a total of 75 yr. Joe aged $\tau_1 = 7$ yr on the outbound trip, and

$$\tau_2 = 2t_1\left[1 - \left(\frac{12}{25}\right)^2\right]^{\frac{1}{2}} = 44 \text{ yr}$$

on the return. The total for Joe is 51 years, so he is younger by 24 years on return.



4-13. *In an inertial frame two events occur simultaneously at a distance of 3 meters apart. In a frame moving with respect to the laboratory frame, one event occurs later than the other by* 10^{-8} *s. By what spatial distance are the two events separated in the moving frame? Solve this problem in two ways: first by finding the Lorentz boost that connects the two frames, and second by making use of the invariance of the spacetime distance between the two events.*

Solution: The interval between the two events

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2$$

must be the same in both frames. In the laboratory frame

$$(\Delta s)^2 = 0^2 + (3 \text{ m})^2 = 9 \text{ m}^2 \ .$$

In the moving frame

$$9 \text{ m}^2 = -\left(3 \times 10^8 \text{ m/s} \cdot 10^{-8} \text{ s}\right)^2 + (\Delta x)^2$$

which gives

$$\Delta x = \sqrt{18 \text{ m}^2} = 4.24 \text{ m} \ .$$

Let (t, x) be coordinates of the inertial frame in which the events are simultaneous, and (t', x') coordinates of a frame moving with respect to this one along the $x-$axis. The Lorentz boost connecting the two frames implies

$$t'_2 - t'_1 = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$

In the unprimed frame, the two events are simultaneous ($\Delta t \equiv t_2 - t_1 = 0$) and separated by $\Delta x \equiv x_2 - x_1 = 3$ m. Then

$$\Delta t' \equiv t_2' - t_1' = -\gamma \frac{v}{c^2} \Delta x = 10^{-8} \text{ s}$$

and solving for γ^2 gives

$$\gamma^2 = 1 + c^2 \left(\frac{\Delta t'}{\Delta x} \right)^2 = 2.$$

Therefore the separation of the two events in the moving frame is

$$\Delta x' = \gamma \Delta x = \sqrt{2}(3) = 4.24 \text{ m}.$$

in one inertial frame then $|\vec{V}| < c$ in any other inertial frame, (b) if $|\vec{V}| = c$ in one inertial frame then $|\vec{V}| = c$ in any other inertial frame, and that (c) if $|\vec{V}| > c$ in any inertial frame then $|\vec{V}| > c$ in any other inertial frame.

4-15

Solution: Orient coordinates so that the x-axis is along V. Then the addition of velocities (4.28a) gives

$$V' = \frac{V - v}{1 - Vv/c^2}$$

Plotting $V'(V)$ for $|V| < c$ we see that it ranges between $-c$ for $V = -c$ and $+c$ for $V = +c$. In all cases $|V'| < c$.

The problem can be analyzed algebraically as will as graphically. The addition of velocities (4.28a) can be written

$$V = \frac{V' + v}{1 + V'v/c^2}$$

For (a), set

$$|V| = \left|\frac{V' + v}{1 + V'v/c^2}\right| < c\,.$$

Solving for V' gives $|V'| < c$. (b) and (c) can be done similarly.