Week 3 – Where we learn about negative pressure!
Week 3 Outline

• Review: Why do $\Omega_0$ and $H_0$ Determine Curvature, $\kappa$ and $R_0$?
• Derive the fluid equation.
• Einstein’s Static Universe
  – Motivation & Introduction of the Cosmological Constant
  – Derive the Acceleration Equation; Define Dark Energy
  – Problems with Einstein’s Static Universe
• Equation of State for the Components of the Universe
• Evolution for each component, $\varepsilon(a)$
  – The dominant component evolves.
• Single-component models
  – Time - Redshift Relation
  – Proper Distances
  – Horizon
• Multi-component models
Modeling the Universe: Friedmann Eqn., Fluid Eqn., and E.O.S.

- What is the connection between $a(t)$ and $\varepsilon(t)$ for any constant $w$?
- Given appropriate boundary conditions, we can solve these equations for $a(t)$, $\varepsilon(t)$, and $P(t)$ for all times.
- Fluid equation tells us how $\varepsilon(t)$ evolves with the expansion described by $a(t)$

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}
\]

\[
\dot{\varepsilon} + \frac{3}{a}(\varepsilon + P) = 0
\]

\[
P = w\varepsilon
\]
The Fluid Equation

\[ \dot{\epsilon} + \frac{3\dot{a}}{a}(\epsilon + P) = 0 \]

\[ P = w\epsilon \]

\[ \frac{\dot{\epsilon}}{\epsilon} = -3(1 + w) \frac{\dot{a}}{a} \]

\[ \epsilon = \epsilon_0 a^{-3(1+w)} \]

... Derive this statement of energy conservation.

... Show that this is a general equation of state.

Together, they tell us how the energy density of a specific component of the universe evolves as the scale factor changes.
Einstein’s Dilemma & The Cosmological Constant

• Soon after the completion of general relativity (1916) people used it to describe the universe.
• However, with only matter there was no way to obtain a static solution, which at that time was the prejudice.
• Einstein added the cosmological constant to his equations to find a static solution…

\[ \Lambda = 4\pi G \rho \]

…reduces the acceleration in Poisson’s equation to zero.

• [Blackboard]

A negative pressure is permissible by the laws of physics. e.g. compress/stretch a piece of rubber.
Even Einstein Makes Mistakes

- The static solution is unstable.
- The required value of the cosmological constant in a static universe is difficult to understand.
- And, when Hubble announced his discovery of the expansion, Einstein’s cosmological constant became unnecessary.

- So the cosmological constant remained on the outskirts of cosmology for a long time.
- Now it’s back because astronomers measure a positive acceleration.
Dynamics of the Universe.

Acceleration equation

- Combine the Friedmann equation and the fluid equation to find out how $a(t)$ changes with time. This convenient form (not independent) is the equation of motion with the second derivative.
- [Blackboard]
Dark Energy

[Note: Includes cosmological constant $w = -1$]

• Inspection of the Acceleration Equation
  – Increasing the mass-energy density slows down the expansion rate of the universe.
  – A positive pressure also slows down the expansion.
  – A universe with $P < -\frac{\varepsilon}{3}$ will accelerate

• A component with $w < -\frac{1}{3}$ is called dark energy.
  – It causes the universe to accelerate (see acceleration eqn.).
  – It has constant energy density during the expansion (see the fluid eqn.).

• A cosmological constant has $P = -\varepsilon$ and is one type of dark energy, which means $w = -1$.

Cosmologists talk in terms of dark energy, instead of cosmological constant, because we don’t know exactly how close $w$ is to $-1$. 
What is Dark Energy?

- Dark energy is interpreted as something with negative pressure filling space.
- Quantum field theory predicts a value of $\varepsilon_{\text{vac}}$ 124 orders of magnitude larger than that measured by astrophysicists.
- Is it some sort of vacuum energy?
- *We really don’t know. But whatever it is, it dominates the dynamics of the universe today!*

What kind of Universe?

- The energy density has many components.
- Dark energy dominates today.
- Let’s look at the numbers.
Matter density of the Universe.

1: Radiation

- Blackbody: \( \rho_{\text{rad}} = a_{\text{rad}} \frac{T^4}{c^2} = 4 \sigma \frac{T^4}{c^3} \), where \( c \) is the speed of light, \( T \) is the temperature of the radiation, \( \sigma \) is the Stefan-Boltzmann constant, \( 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \)

- So \( \rho_{\text{rad}} = 4.6 \times 10^{-31} (T/2.725\text{K})^4 \text{ kg/m}^3 = 4.6 \times 10^{-34} \text{ g/cm}^3 \)

- It is convenient to write this down in terms of the critical density, the amount of energy/matter needed to “close” the universe, \( \rho_{\text{crit}} = \frac{3H_0^2}{8\pi G} = 9.5 \times 10^{-27} \text{ kg/m}^3 \) or \( 9.5 \times 10^{-30} \text{ g/cm}^3 \).

- The density of radiation is \( 4.8 \times 10^{-5} \rho_{\text{crit}} \)

- This can be written as \( \Omega_{\text{rad}} \sim 5 \times 10^{-5} \)
Matter density of the Universe.

2: Neutrinos

- Limits on neutrino mass density come from:
  - Oscillations (lower limit; superkamiokande)
  - Large scale structures (upper limits; CMB+2dF; Sanchez et al. 2006)
  - Cosmic rays striking atmosphere

- In critical units neutrino mass density is between:
  $0.0010 < \Omega_\nu < 0.0025$
Matter density of the Universe.

3: Baryons

- People have counted the amount of mass in visible baryons.

- Baryonic inventory (total=0.045±0.003 from nucleosynthesis and CMB):
  - Stars $\Omega_* = 0.0024 \pm 0.0007$ (comparable mass in neutrinos and stars!)
  - Planets $\Omega_{\text{planet}} \sim 10^{-6}$
  - Warm intergalactic gas 0.040±0.003

- Most of baryons are in intergalactic medium, filaments the cosmic web.


Matter density of the Universe.

4: Dark matter

• Dark matter is harder to count, because we can only “see” it via its gravitational effects
• One way to count it is for example is to measure the dark matter to baryon ratio in clusters
• Assume that this number is representative of the Universe because the collapsed volume is large
• Take the fraction of baryons (from BBN) and multiply
• This and other methods give $\Omega_{dm}=0.23$
• The total amount of matter is given by: $\Omega_m=\Omega_{dm}+\Omega_b=0.27$
Matter density of the Universe.

5: Dark energy (or $\Lambda$)

- As we will see most of the energy in the universe appears to be of a mysterious form called dark energy.
- Dark energy repels instead of attracting, and therefore causes the expansion of the universe to accelerate.
- One form of dark energy is the cosmological constant ($\Lambda$), introduced by Einstein a long time ago, and this is a purely geometrical term... We will explain this later.
- According to current measurements $\Omega_{\text{de}} \sim 0.72$ or $\Omega_\Lambda \sim 0.72$. 
Evolution of the Energy Density

Fluid Equation

\[
\dot{\epsilon} + \frac{3\dot{a}}{a} (\epsilon + P) = 0
\]

General Equation of State

\[
P = w\epsilon
\]

\[
\frac{\dot{\epsilon}}{\epsilon} = -3(1 + w)\frac{\dot{a}}{a}
\]

\[
\epsilon = \epsilon_0 a^{-3(1+w)}
\]

Together, they determine how the energy density evolves, i.e., \( \epsilon(a) \).
What kind of Universe? Who rules?

- Energy density in matter is much larger today than that in CMB photons and neutrinos.
- Depending on $w$, energy density evolves at different rates.
- The dominant species changes with time. **Radiation was once the dominant component.**
- Set $\varepsilon_{\text{rad}}(a_{\text{rm}}) = \varepsilon_{\text{m}}(a_{\text{rm}})$, and solve for $a_{\text{rm}}$.

What redshift corresponds to $a_{\text{rm}}$? Recall, $1 + z = 1 / a(t_e)$. 
When would the cosmological constant be the dominant energy density?

- The cosmological constant dominates the energy density in Friedmann’s equation today. We have \( \frac{\epsilon_{\Lambda,0}}{\epsilon_{m,0}} \sim \frac{0.7}{0.3} = 2.3 \).

- The energy density of the cosmological constant is time independent.

- So dark matter dominated at some point in the past. Why?

Could you calculate the scale factor, \( a_{\Lambda m} \), when \( \epsilon_{\Lambda}(a_{\Lambda m}) = \epsilon_{m}(a_{\Lambda m}) \)?
Quiz #5

- Using the following parameters for the concordance cosmology,
  - \( \Omega_{m,0} = 0.3 \) where \( \varepsilon_{0,m} = \Omega_{m,0} \varepsilon_{\text{crit},0} \)
  - \( \Omega_{\Lambda,0} = 0.7 \), and
  - \( \Omega_{\gamma} = 8 \times 10^{-5} \),

answer the following questions.

1. Find the redshift of matter-radiation equality.

2. Find the redshift where dark energy became the dominate form of the energy density.
Quiz #5

- Using the following parameters for the concordance cosmology,
  - $\Omega_m = 0.3$,
  - $\Omega_\Lambda = 0.7$, and
  - $\Omega_\gamma = 8 \times 10^{-5}$,

answer the following questions.

1. Find the redshift of matter-radiation equality.
   
   **(Check Yourself)**
   
   Answer: $1 + z_{m\gamma} = 3600$

2. Find the redshift where dark energy became the dominate form of the energy density.
   
   **(Check Yourself)**
   
   Answer: $z_{m\Lambda} = 0.32$
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“Concordance cosmology” or “Benchmark model”

- Our current best guess
- Photons and neutrinos
- Matter (baryonic and dark)
- Cosmological Constant
- Spatially Flat
Modeling the Universe: Friedmann Eqn., Fluid Eqn., and E.O.S.

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\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0
\]

\[
P = w\varepsilon
\]
Generalized Friedmann Equation

\[ H^2(t) = H_0^2 \left[ \Omega_{\gamma,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\kappa}a^{-2} + \Omega_{\Lambda} \right] \]

\[ H^2(t) = H_0^2 \left[ \Omega_{\gamma,0}(1 + z)^4 + \Omega_{m,0}(1 + z)^3 + \Omega_{\kappa}(1 + z)^2 + \Omega_{\Lambda} \right] \]

- Solution is straightforward by numerical integration.
- Let’s examine properties of the concordance model.
Concordance cosmology.

\( a(t) \) from numerical integration

- Evolved from radiation dominated to matter dominated, and is now entering a phase dominated by dark energy
- Current age is 13.5 Gyr
Single component Universes. Examples

• Let’s start simple… an empty Universe..
• Solve the Friedmann Equation..

[Black board]

• Let’s compare a flat universe which contains only a cosmological constant.
• Solve the Friedmann Equation..

[Black board]

• Let’s get more general. Solve for all flat models with a single component, excluding \( w = -1 \).
• Solve the Friedmann Equation..

[Black board]

Quiz 6 – Answer in Gaucho Space
Problem 1: What is the value of \( w \) in this model?
Problem 2: What is the value of \( \kappa \) in this model?
Solutions to the Friedmann Eqn. for “Flat” Universe with a single “fluid”

- When curvature constant = 0, the universe has Infinite Volume

- Solution is power law in time: 
  \[ a(t) = \left( \frac{t}{t_0} \right)^{2/(3+3w)} \]
  where 
  \[ t_0 = \frac{1}{(1+w)} \left\{ \frac{c^2}{(6\pi G\epsilon_0)} \right\}^{0.5} \]

- This solution is not valid when \( w = -1 \).

- Plot of \( a(t) \) to build intuition.
  - Matter only: \( a(t) \) scales as \( t \) to the power of \( 2/3 \)
  - Radiation only: \( a(t) \) scales as \( t \) to the power of \( 1/2 \)
  - \( \Lambda \) only: \( a(t) \) exponential in \( t \)

- Will these universes expand forever?
Properties of Flat, Single Component Universes.

1. The Empty Universe
   • Expansion or contraction rate is constant; this means the age is equal to the Hubble time.
   • This universe has negative curvature. What is the radius of curvature?
   • Redshift-time relation is linear.
   • Redshift-distance relation; can see arbitrarily far.
     • Why can you see further than c/H₀?
     • Objects with high redshifts are seen as they were when the universe was very young, and their proper distance was small
Properties of Flat, Single Component Universes.

2. Cosmological Constant (w = -1)

- The cosmological constant is one candidate for the dark energy that people talk about today. In the Benchmark Model, the cosmological constant dominates the energy density at late times.
- What happens for a pure cosmological constant?
  - The age is infinite.
  - Expansion rate is constant.
  - Horizon is infinite.
“Flat” Universe with a single “fluid.”

Redshift - Time Relationship

- \[1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}\]
- Age \[t_0 = \frac{2}{3} H_0^{-1} / (1 + w)\]
  - \[t_0 H_0 = 2/3\) (matter only)
  - \[t_0 H_0 = 1/2\) (radiation only)
- Lookback time = \[t_0 - t_e\]

- The age of the universe is less than a Hubble time for \(w > -1/3\).
- Is the matter-only or radiation-only model older?
“Flat” Universe with a single “fluid.”
Redshift - Distance Relationship

\[ d_p(t_0) = \frac{c}{H_0} \cdot \frac{2}{1 + 3w} \left[ 1 - \left(1 + \frac{z}{1 + 3w} \right)^{\frac{1 + 3w}{2}} \right] \]

- Flat, Matter only universe has a maximum \( d_p(t_e) = \left(\frac{8}{27}\right) \frac{c}{H_0} @ z = \frac{5}{4} \)
- Flat, Radiation only has a maximum at \( z = 1 \)
- Empty universe had a maximum at \( z = 1.7 \)
Review Distances

- Has a finite horizon of 14000 Mpc

- $d_p(t_e)$ has a maximum at $z=1.6$

**FIGURE 6.6** The proper distance to a light source with observed redshift $z$. The upper panel shows the distance at the time of observation; the lower panel shows the distance at the time of emission. The bold solid line indicates the Benchmark Model, the dot-dash line a flat, lambda-only universe, and the dotted line a flat, matter-only universe.
Generalized Friedmann Equation

\[ H^2(t) = H_0^2 [\Omega_{\gamma,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_\kappa a^{-2} + \Omega_\Lambda] \]

\[ H^2(t) = H_0^2 [\Omega_{\gamma,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_\kappa (1+z)^2 + \Omega_\Lambda] \]

• You work on some special cases that yield an analytic solution in HW #3 and HW #4.
• Let’s examine some of the properties of these model universes.
• Why? Because these models make predictions that can be refuted or verified!
1. Matter + Curvature Models

- Consider all values of $\Omega_m$ ($\Omega_\Lambda = 0$ and $\Omega_\gamma = 0$)
  - Allows closed, flat, or open geometry (i.e., positive, zero, or negative curvature, $\kappa = +1, 0, \text{or} -1$)
  - This is still a small subset of all possible models!
  - New phenomena: Maximum scale factor when $\kappa = +1$.

Curvature + Matter
Fate of Matter + Curvature Models

- **Density determines destiny (in these models only).**
  -- The matter density alone ($\Omega_m$) determines whether the expansion stops.
  -- It is equivalent to think in terms of curvature or matter density in these models.

- When $\Omega_m > 1$, the contraction time is equal to the expansion time; the solution is symmetric in time.

- When do these solutions look like the matter only solution?
Age of the Universe (HW #3, Prob. 3)

- A matter + curvature universe has finite duration.
  - The more matter there is, the shorter the age of the universe.
  - The 4.6 Gyr age of the solar system clearly rules out matter + curvature models with $\Omega_m \gg 1$.

Notice that in matter + curvature models $\Omega_0$ is $\Omega_m$. 

![Graph showing the age of the solar system vs. $\Omega_0$]
Week 3 Summary

• Derive the fluid equation.
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  – Motivation & Introduction of the Cosmological Constant
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Elbbuh Niwde’s Discovery (HW #4, Prob. 2)

- You are asked to consider observations in a matter + curvature universe that is contracting, $\Omega_0 > 1$.
- Why does Dr. Niwde observes that all the galaxies are moving towards her?

Given measurements of $H_0$ and $\Omega_0$, how much time remains before the Big Crunch? i.e., What is $t_{\text{crunch}} - t_o$?
Matter + Lambda + Flat Models

- $\Omega_m + \Omega_\Lambda = 1$
- Still a small subset of all models.
- A positive cosmological constant introduces new phenomena.
  - Most models expand forever, a.k.a. Big Chill
  - Many (but not all) models accelerate
- We can get an age $t_0 = 0.964H_0^{-1} = 13.5$ Gyr using the measured values $\Omega_{0,m}=0.3$ and $\Omega_\Lambda=0.7$. 

![Graph showing cosmological constant vs. matter density]

![Graph showing expansion of the universe with different cosmological constants]
Dark Energy + Curvature (HW #4, Prob. 3)

• Consider an expanding positively curved universe with $\Omega_\Lambda > 1$.

• This universe had no Big Bang. You are asked to show that this universe underwent a Big Bounce at a scale factor

$$a_{\text{bounce}} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}$$

• What observations could be made to determine whether we live in such a universe?
  – Measure matter content, $\Omega_0$
  – Measure maximum redshift for galaxies (i.e., minimum $a(t_e)$).

Note: Curve shown for $\Omega_0 = 0.3$. 

FIGURE 6.4 The scale factor $a$ as a function of $t$ in four different universes, each with $\Omega_{m,0} = 0.3$. The dashed line shows a “Big Crunch” universe ($\Omega_{\Lambda,0} = -0.3$, $\kappa = -1$). The dotted line shows a “Big Chill” universe ($\Omega_{\Lambda,0} = 0.7$, $\kappa = 0$). The dot-dash line shows a loitering universe ($\Omega_{\Lambda,0} = 1.7134$, $\kappa = +1$). The solid line shows a “Big Bounce” universe ($\Omega_{\Lambda,0} = 1.8$, $\kappa = +1$).