Physics 220: Problem Set 2 - solution
due April 24, 2012.

1. Show that the longitudinal susceptibility, $\chi_l = \partial\langle S_n^x\rangle/\partial h_\perp$, of the 1d transverse field Ising chain has a logarithmic divergence at the quantum critical point.

   We wrote down the formula for this susceptibility (not sure why I am calling it longitudinal. Probably transverse would be better?) in class (it is the same as the second derivative of the energy):

$$\chi_l = \int_0^\pi \frac{dk}{2\pi} \frac{2J^2 \sin^2 k}{(J^2 + 4h_\perp^2 - 4h_\perp J \cos k)^{3/2}}. \tag{1}$$

   The critical point occurs when $h_\perp = J/2$. We see that at that precise point, the denominator vanishes when $k = 0$, and the integral becomes divergent. So we let $h_\perp = J/2(1+t)$, with $|t| \ll 1$, which describes the vicinity of the critical point, and approximate $\cos k \approx 1 - k^2/2$, $\sin k \approx k$. This gives, keeping leading terms

$$\chi_l \approx \frac{1}{\pi J} \int_0^\pi dk \frac{k^2}{(t^2 + k^2)^{3/2}}. \tag{2}$$

   Writing $k = |t|x$, we have

$$\chi_l \approx \frac{1}{\pi J} \int_0^{\pi/|t|} dx \, \frac{x^2}{(x^2+1)^{3/2}}. \tag{3}$$

   At small t, we see that this is controlled by $x \gg 1$, so can approximate it by

$$\chi_l \approx \frac{1}{\pi J} \int_1^{\pi/|t|} \frac{dx}{x} \sim \frac{\ln(\pi/|t|)}{\pi J}. \tag{4}$$

   Technically, this is correct only up to constant terms in t, i.e. only the coefficient of the log is correct, and the π inside the $\ln$ is not reliable. But this is enough for us.

2. For the 3d Ising model (or the 2d quantum transverse field Ising model), the critical point is described by a scale invariant field theory with an "energy density" operator ε and a spin operator σ, just as in 2d (1d quantum), but with $d_\epsilon = 1.59$ and $d_\sigma = 0.52$. Find the specific heat exponent α, the order parameter exponent β, and the correlation length exponent ν.

   The effective field theory for the vicinity of the critical point is

$$F = \int d^3x \, \{t\varepsilon - h\sigma\}, \tag{5}$$

   where $t \sim (T - T_c)/T_c$ is the deviation from the critical temperature, and h is proportional to the magnetic field. Under a scale transformation, $x \to bx$, we have $t \to b^{3-d_\varepsilon} t$ and $h \to b^{3-d_\sigma} h$.

   - To get the specific heat exponent, we need the internal energy density at zero field. This is

$$u = \langle\varepsilon\rangle \sim b^{-d_\varepsilon} f(b^{3-d_\varepsilon} t) \sim |t|^{d_\varepsilon/(3-d_\varepsilon)}, \tag{6}$$

     choosing $b = |t|^{-1/(3-d_\varepsilon)}$. The specific heat is $\partial u/\partial t \sim |t|^{d_\varepsilon/(3-d_\varepsilon)-1} \equiv |t|^{-\alpha}$, so

$$\alpha = (3 - 2d_\varepsilon)/(3 - d_\varepsilon). \tag{7}$$

     If you take $d_\varepsilon = 1.59$, you get $\alpha = -0.13$. Actually, I gave the wrong number in the homework. It is really $d_\varepsilon = 1.41$, which gives $\alpha = 0.11$.

- Again we take zero field, and write

$$m = \langle \sigma \rangle \sim b^{-d_\sigma} f(b^{3-d_\varepsilon} t) \sim |t|^{d_\sigma/(3-d_\varepsilon)}. \tag{8}$$

  By definition, $m \sim |t|^\beta$, so

$$\beta = d_\sigma/(3 - d_\varepsilon). \tag{9}$$

  Using the values given in the homework, we get $\beta = 0.369$, but if we use the corrected value, we get $\beta = 0.328$.

- Finally, for the correlation length, since it is a length,

$$\xi \sim bf(b^{3-d_\varepsilon} t) \sim |t|^{-1/(3-d_\varepsilon)}, \tag{10}$$

  which, since $\xi \sim |t|^{-\nu}$ gives

$$\nu = 1/(3 - d_\varepsilon). \tag{11}$$

  This gives, using the value from the homework, $\nu = 0.709$, or with the correct value, $\nu = 0.63$.