2 The Dirac equation.

So far we have dealt with spinor fields $\psi_a$, $\bar{\psi}^\dot{a}$, and the existence of invariants $\sigma_{ab}^\mu$ and $\bar{\sigma}^{\dot{a}b\mu}$, $\epsilon_{ab}$.

Now we want to study covariant equations of motion for spinor fields. Just like for bosons we have the Klein-Gordon equation $(\partial_\mu \partial^\mu - m^2)\phi = 0$, we can have the same equation for spinors. However, we can also consider forming covariant equations with one derivative.

For example, we can consider the following object

$$i\bar{\sigma}^{\mu\dot{a}b}\partial_\mu \psi_b$$

This object transforms exactly like $\psi^\dagger \dot{a}$, so we can make an equation of the form

$$i\bar{\sigma}^{\mu\dot{a}b}\partial_\mu \psi_b = m \psi^\dagger \dot{a}$$

Similarly, we can consider the complex conjugate of this equation, and we can raise and lower the indices as appropriate, so that

$$i\bar{\sigma}^{\mu} \partial_\mu \psi^\dagger \dot{b} = m^* \psi_a$$

Notice that by changing the complex phase of the field $\psi$, the field $\psi^\dagger$ transforms with the opposite phase. From here, we find that it is possible to choose $m$ so that it is real $m = m^*$.

We can put these two equations together as follows

$$\begin{pmatrix} m \delta^b_a & -i\sigma^{\mu}_{ab} \partial_\mu \\ -i\bar{\sigma}^{\dot{a}b\mu} \partial_\mu & m \delta^\dot{a}_b \end{pmatrix} \begin{pmatrix} \psi_b \\ \psi^\dagger \dot{b} \end{pmatrix} = 0$$

We want to develop a notation where we can suppress all those spinor indices. Thus we will have $\psi$ always be a Weyl spinor with an un-dotted index in the lower position. Similarly, $\psi^\dagger$ will be a spinor with an upper dotted Weyl index.

Similarly $\sigma^\mu$ will have two lower spinor indices, one undooted, and one dotted in that order, and $\bar{\sigma}^{\dot{a}b\mu}$ will have two upper spinor indices with the first one dotted and the second one un-dotted.

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With these conventions, we could write

$$i\sigma^\mu \psi = m\psi^\dagger$$

and it encapsulates the equation 2 with all the indices contracted properly.

With these conventions, all index contractions are in the natural order for matrix multiplication.

We will also have the convention

$$\chi \psi = \chi^a \psi_a$$

and for it’s complex conjugate

$$\psi^\dagger \chi^\dagger = \psi^\dagger_\dot{a} \chi^\dagger_{\dot{a}}$$

We will also introduce the $4 \times 4$ matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\dot{a}b} \\ \sigma^{\mu\dot{a}b} & 0 \end{pmatrix}$$

and we can consider the field

$$\Psi = \begin{pmatrix} \psi_a \\ \psi^{\dagger}_{\dot{a}} \end{pmatrix}$$

With these conventions, our covariant spinor equation reads

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0$$

This is called the Dirac equation.

Multiplying on the left by $(i\gamma^\mu \partial_\mu + m)$, it is easy to prove using the identities that the sigma matrices satisfy, that if $\Psi$ is a solution of the Dirac equation, then it is also a solution of the Klein-Gordon equation.

$$(\partial_\mu \partial^\mu + m^2)\Psi = 0$$

This follows from

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

after some manipulations.
Notice that we should count $\Psi$ above as a real field. This is because the complex conjugate of $\Psi^*$ is proportional to $\Psi$ itself. The constant of proportionality involves raising and lowering the indices with the $\epsilon_{ab}$ tensor and $\epsilon^{ab}$, as follows

$$\Psi^* = \begin{pmatrix} 0 & \epsilon \\ \epsilon^T & 0 \end{pmatrix} \Psi = \tilde{C}\Psi$$

where $\tilde{C}$ is related to the charge conjugation matrix. Numerically it is identical to $\gamma^0$, but it has a different index structure in terms of spinor indices, so it is best not to confuse them (although in practice this does not make much difference at the level of performing computations, conceptually the two are very different objects).

It is also convenient to introduce the following object

$$\bar{\Psi} = \Psi^T \beta$$

The definition of $\beta$ is designed so that

$$\bar{\Xi} \Psi = \xi \psi + \xi^\dagger \psi^\dagger$$

which is a natural scalar. With these conventions, we have that

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

The charge conjugation matrix takes the form

$$\Psi = C\bar{\Psi}^T = \tilde{C}\Psi^* = C\beta \Psi^*$$

This derivation of the Dirac equation shows that it is automatically relativistically covariant.

It would be nice if we could derive it from an action principle.

The obvious choice would involve natural scalars under the Lorentz group, built out of $\psi$ and $\psi^\dagger$.

For example, consider a term of the form

$$m\psi^a \psi_a$$

If the $\psi$ are ordinary numbers, then we would have that

$$m\psi = m\epsilon^{ab} \psi_b \psi_a = -m\psi_b \epsilon_{ba} \psi_a = -m\psi \psi^b = -m\psi^b \psi_b = -m\psi \psi = 0$$
Thus it would seem at first sight that we can not derive the Dirac equation from a natural action principle.

However, we would want that $\psi \psi$ is not equal to zero. In order to have that, we can introduce the notion of anticommuting classical numbers. These are sometimes called a-numbers, or Grassman numbers.

Ordinary commuting numbers are called c-numbers. Eventually all fields are going to become operators, so having operators that anticommute is rather natural.

The spinor fields are going to be a-numbers. Thus, the following equations hold

$$\psi_a \chi_b = -\chi_b \psi_a$$

(20)

With these conventions, objects such as $\psi^a \psi_a$ are non-zero. The extra minus sign from the anticommutation relation above cancels the minus sign obtained from the fact that $\epsilon^{ab}$ is antisymmetric.

Our naive Dirac action would be

$$\psi^\dagger i\sigma^\mu \partial_\mu \psi + tm\psi \psi + tm\psi^\dagger \psi^\dagger$$

(21)

where $t$ needs to be normalized to reproduce the Dirac equation. However, when we take derivatives with respect to $\psi^\dagger$, the term $\psi^\dagger \psi^\dagger$ is a square, so we would get a factor of 2. In this case, we find that $t = 1/2$. This factor of 1/2 is the same as that appearing for a real scalar field kinetic term and mass term.

If we used Dirac’s spinor notation with four component spinors, we would get

$$\frac{1}{2} \bar{\Psi} \gamma^\mu i \partial_\mu \Psi + \frac{1}{2} m \bar{\Psi} \Psi$$

(22)

Remember that $\Psi$ is in this case a real field, as

$$\Psi = \bar{C} \Psi^*$$

(23)

A Dirac spinor with this reality constraint is called a Majorana spinor.

If we remove this restriction, we get that the four components of a Dirac spinor are independent complex numbers, and we can decompose a Dirac spinor into two Majorana spinors : the real and imaginary part of the Dirac spinor.

For a complex field, we use the same conventions for spinors than for scalars, so that the kinetic term and mass term has no factor of 1/2, and all four complex components of the spinor are treated independently.
2.1 Solving the Dirac equation

First, the Dirac equation has no explicit coordinate dependence. This is, it is translation invariant. Moreover, it is a linear partial differential equation. Thus, we can use the superposition principle to solve the problem.

Because of translation invariance, we can use plane wave solutions of the form

$$\Psi \sim u(p) \exp(ipx)$$  \hspace{1cm} (24)

Remember that in our conventions, positive energy corresponds to behavior of the form $\exp(-iEt)$. Let us study first this situation.

Because the Dirac equation implies the Klein Gordon equation, we need to satisfy $p^2 + m^2 = 0$. Since $m$ is real, this implies that $p^2$ corresponds to a future pointing object. In particular, we can go to a reference frame where $\vec{p} = 0$, and $p^0 = m = E$.

In this reference frame, the Dirac equation takes the following form

$$\begin{pmatrix} -m & 0 & E & 0 \\ 0 & -m & 0 & E \\ E & 0 & m & 0 \\ 0 & E & 0 & -m \end{pmatrix} u(p) = 0$$  \hspace{1cm} (25)

Notice that the Dirac equation splits, and we have two solutions

$$u_+(p) \sim \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(p) \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (26)

These are distinguished by the spin in the $z$ direction. We normalize these with an extra factor of $\sqrt{m}$.

Similarly, we can consider solutions with negative energy. These will be

$$v_+ = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_- = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (27)

Now, remember that for a Majorana field, we have that $\tilde{C} \Psi^* = \Psi$. 


Since by convention we have that \( \psi \) and \( \psi^\dagger \) have upper and lower indices contracted, we find that the reality conditions are satisfied if we use solutions of the form

\[
u_+(0) \exp(-imt) + v_+(0) \exp(imt) \quad (28)
\]

This is, \( \tilde{C}u_+^*(p) = v_+(p) \), equivalently, \( C\bar{u}_s(p)^T = v_s(p) \).

### 2.2 Gamma matrix technology

Here we will follow very closely the chapters 38 and 47 of M. Srednicki. We need also the definition

\[
\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \sim i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (29)
\]

This is so that the Weyl spinors are Dirac spinors projected with \((1 \mp \gamma_5)/2\)