Week 14: Group theory primer [1]

Useful Reading material

- *Fulton and Harris, Representation Theory, Graduate texts in Mathematics, Springer*

1 SU(N)

Most of the analysis we are going to do is for $SU(N)$. So we will mention some modifications accordingly, but without too many details.

We assume that Lie algebras are defined by generators T^a, satisfying the equations

$$[T^a, T^b] = if^{abc}T^c \tag{1}$$

where F^{abc} are the structure constants of the Lie algebra. A representation of the Lie algebra is a realization of these commutation relations on a set of $M \times M$ matrices.

The theory of the group $SU(N)$ begins with a Hilbert space where $SU(N)$ acts as a set of isometries. We have a basis of N-orthogonal vectors $|1\rangle, \dots |N\rangle$. We are going to define an ordering on the basis elements. This ordering is not an ordering on the Hilbert space.

IN the ordering we have

$$|1\rangle > |2\rangle > \dots |N\rangle \tag{2}$$

At this stage this ordering is completely arbitrary. Given this basis, we can define the Cartan elements of the Lie algebra of $SU(N)$ as those that are diagonal in the $|i\rangle$ basis. This is, the $|i\rangle$ are eigenvectors of the Cartan. In general, the Cartan is a maximal set of linearly independent commuting generators of the Lie algebra. Since they commute, they can always be diagonalized simultaneously.

Remember that the $SU(N)$ generators are traceless (this is part of the definition os $SU(N)$). There are $N-1$ linearly independent diagonal traceless matrices. Thus the Cartan subalgebra has $N-1$ generators. Let us call them H_i. Since the $|j\rangle$ are eigenvectors, we have the relation

$$H_i|j\rangle = \lambda_{ij}|j\rangle \tag{3}$$

The numbers λ_{ij} are called the weights of the vector $|j\rangle$. The state $|1\rangle$ will be called the highest weight state.

It is natural to split thee other generators of the lIe algebra in terms of their eigenvalues with respect to the H generators. This is, we diagonalize the action of the H on the Lie algebra itself.

We will define the roots of the Lie algebra $\tilde{E}_\alpha$ by the relation

$$[H_i, \tilde{E}_\alpha] = \alpha_i \tilde{E}_\alpha \tag{4}$$

Notice that the $\tilde{E}_\alpha$ are not hermitian. Indeed, if the H are hermitian, then

$$[H_i, \tilde{E}_\alpha^\dagger] = -([H_i, E_\alpha])^\dagger = -\alpha_i \tilde{E}_\alpha^\dagger \tag{5}$$

So we can have the relation

$$\tilde{E}_\alpha^\dagger = \tilde{E}_{-\alpha} \tag{6}$$

The list of the α_i are called the weights of $\tilde{E}_\alpha$. We use the α_i themselves as the labels representing α

Using the Jacobi identity, we can calculate the structure constants in this basis

$$\begin{aligned} [H - i[\tilde{E}_\alpha, \tilde{E}_\beta]] &= [[H_i, \tilde{E}_\alpha], \tilde{E}_\beta] + [\tilde{E}_\alpha, [H_i, \tilde{E}_\beta]] && (7) \\ &= (\alpha_i + \beta_i)[\tilde{E}_\alpha, \tilde{E}_\beta] && (8) \end{aligned}$$

So we have that

$$[\tilde{E}_\alpha, \tilde{E}_\beta] = s_{\alpha\beta} \tilde{E}_{\alpha+\beta} \tag{9}$$

because, after all, $[\tilde{E}_\alpha, \tilde{E}_\beta]$ is an eigenvector of weight $\alpha + \beta$. Thus, the labels α, β can be added or subtracted by commutation (use $\tilde{E}_{-\alpha}$ for substraction).

If we have any representation of G, we always get some collection of weights γ_i under the Cartan. Thus we have a basis classified by the eigenvalues (weights)

$$H_i|\gamma\rangle = \gamma_i|\gamma\rangle \tag{10}$$

From here, we have that

$$\begin{aligned} H_i \tilde{E}_\alpha|\gamma\rangle &= [H_i, \tilde{E}_\alpha]|\gamma\rangle + \tilde{E}_\alpha H_i|\gamma\rangle && (11) \\ &= \alpha_i (\tilde{E}_\alpha|\gamma\rangle) + \tilde{E}_\alpha \gamma_i|\gamma\rangle && (12) \\ &= (\alpha_i + \gamma_i)\tilde{E}_\alpha|\gamma\rangle \simeq n_{\alpha\gamma}|\gamma + \alpha\rangle && (13) \end{aligned}$$

This is, the $\tilde{E}_\alpha$ raise or lower the weights by the labels α. *They are generalized ladder operators.*

A weight will be called positive if on the basis $|1\rangle, \dots |N\rangle$ we have that

$$\tilde{E}_\alpha |m\rangle > |m\rangle \tag{14}$$

In standard conventions where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \dots \tag{15}$$

The $\tilde{E}_\alpha > 0$ are upper triangular matrices with only one non-zero entry.

A positive root E_α is called a simple root if it raises weights by a minimal amount. These are written without a tilde.

This is, one such that $E_\alpha |m+1\rangle = |m\rangle$ for some m. There are $N-1$ such simple roots. As a set they hug the principal diagonal of the $N \times N$ matrices, on the upper triangular side. These roots can be labeled as $E_{12} \simeq E_1, E_{23} \simeq E_2 \dots$, where on the left we have the corresponding nonzero component and on the right we just have the first label.

For a general simple Lie algebra all the same concepts apply. One can prove that there are always $\dim(H)$ simple roots. (We always need at least $\dim(H)$ independent elements to distinguish the H_α by their eigenvalues on *Lie*).

It's also obvious that

$$[E_\alpha, E_{-\alpha}] \in H \tag{16}$$

Since we have $\dim(H)$ simple roots α, the right hand side must be a basis. Thus, define the Cartan elements by

$$H_\alpha = [E_\alpha, E_{-\alpha}] \tag{17}$$

Each of these triples $E_\alpha, H_\alpha, E_{-\alpha}$ defines an $SU(2)$ algebra. Normalize the constants so that

$$[H_\alpha, E_\alpha] = 2E_\alpha \tag{18}$$

$$[H_\alpha, E_{-\alpha}] = -2E_{-\alpha} \tag{19}$$

This is called the Chevaley basis. It is constructed so that all of the weights of the roots are integer valued.

Given the set $\{H_\alpha, E_\alpha, E_{-\alpha}$ one can always generate the the full Lie algebra by commutators. For example, in $SU(2)$, we have that

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_+^\dagger \tag{20}$$

and

$$[E_+, E_-] = H = \sigma^3 \tag{21}$$

Thus, in the spin 1/2 representation, the H is twice the usual σ_z. This means that H is normalized to twice the spin (which is always an integer).

In general, we have relations of the form

$$[H_\beta, E_\alpha] = c_{\beta\alpha} E_\alpha \tag{22}$$

The $c_{\beta\alpha}$ must be some important matrix describing the Lie algebra. These are integers, and this is called the Cartan Matrix. By construction $C_{\alpha\alpha} = 2$.

For simply laced groups, the Cartan is symmetric. (These are the ADE groups: look at a table)

If we insist on unitary representations of $SU(N)$ (as required in most applications), then all the $SU(N)$ have finite dimensional representations, and one can not raise thee weights indefinitely.

A state is called a highest weight state if it is annihilated by all the positive (simple roots)

$$E_\alpha |W\rangle = 0 \forall \alpha > 0 \tag{23}$$

The weights are given by

$$H_\alpha |W\rangle = w_\alpha |W\rangle \tag{24}$$

the w_α are not arbitrary (they end up being integers, and are highest weight states for a list of $SU(2)$).

Thus the allowed set of weights forms a lattice of dimension $\dim H$. (Because we can tensor representations we can add weights). This is called the weight lattice (denoted by Λ_W). The sublattice generated by the roots is called the root lattice (Λ_r). These are in general different lattices.

For $SU(N)$ the Cartan Matrix can be calculated to be given by

$$c_{\alpha\beta} \sim \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{25}$$

These are the weights of the simple roots with respect to the Cartan. The different $SU(2)$ subalgebras act onthe spaces generated by $|i\rangle, |i+1\rangle$ as doublets.

We can draw this matrix in a convenient graphical notation, where for each root (or each H_α) we draw a node. We then draw a line for each pair of nodes that is connected by a -1. This graphical representation is called a Dynkin diagram. The ADE groups have a similar structure, with -2 in the diagonal, and -1 in various other places. For non-simply laced groups the rules are slightly more complicated (there are long root and short roots, so some elements of the diagonal are not of size 2). You should read abut these separately.

For example, we can have for $SU(6)$

$$\bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \tag{26}$$

The classification of simple Lie algebras is equivalent to the classification of Dynkin diagrams. The $C_{\alpha\beta}$ is always a positive definite matrix or Lie algebras.

To describe representations, we can label the Dynkin diagram with the weights of the highest weight state. For example, the fundamental of $SU(N)$ is labeled by

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \\ \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \end{array} \tag{27}$$

The symmetric tensor representation is labeled by

$$\begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ \\ \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \end{array} \tag{28}$$

And the antisymmetric tensor representation is labeled by the highest weight state

$$\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ \\ \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \text{---} & \bullet \end{array} \tag{29}$$

We can also label the Dyknin diagram for arbitrary weights (not a highest weight state).

For the fundamental of $SU(N)$, these are given by

$$\begin{array}{ccccccccc} 0 & & 0 & & -1 & & 1 & & 0 \\ \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \end{array} \tag{30}$$

where a -1 is followed by a 1. This is easy to show with the specific matrices that we have given so far.

The complex representation is characterized by Charge conjugation, so all weights change sign. This way we find that the highest weight state of the antifundamental is given by

$$\begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 1 \\ \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \end{array} \tag{31}$$

as this is the only purely positive eight that is allowed. This is the same as the totally antisymmetric representation with $N-1$ indices.

1.1 Tensor products of representations

Let R, R' be two unitary representations of a Lie algebra G. These are classified by their highest weight states, which we shall denote by $hws(R)$ and $hws(R')$.

The spaces R, R' are two Hilbert spaces. Their tensor product is denoted by $R \otimes R'$ and this is a vector space whose elements are by definition of the form

$$\sum |v\rangle \otimes |v'\rangle \tag{32}$$

with $|v\rangle \in R$ and $|v'\rangle \in R'$.

The norm is the obvious norm, given by

$$(\langle\tilde{v}| \otimes \langle\tilde{v}'|)|v\rangle \otimes |v'\rangle = \langle\tilde{v}|v\rangle\langle\tilde{v}'|v'\rangle \tag{33}$$

with the usual linearity axioms.

If G acts on R by unitary transformations, and on R' by unitary transformations, then the action of U_R, $U_{R'}$ preserves the norm in the Hilbert space.

One can consider the action of G on $R \otimes R'$ given by

$$U_{R\otimes R'} = U_R \otimes U_{R'} \tag{34}$$

so that

$$U_{R\otimes R'}|v\rangle \otimes |v'\rangle = U_R|v\rangle \otimes U_{R'}|v'\rangle \tag{35}$$

It is easy to check the group properties, and that $U_{R\otimes R'}$ preserves the norm.

The group G is continuous, which means that U can be made infintesimal, and then we have the equation

$$U_\theta \sim 1 + i\theta^a T^a \tag{36}$$

where the T^a are the generators of the Lie algebra. Applying the distributivity and to linear order in θ we find that

$$\begin{aligned} U_{R\otimes R';\theta} \simeq U_{R;\theta} \otimes U_{R';\theta} &= (1 + i\theta^a T^a_R) \otimes (1 + i\theta^a T^a_{R'}) && (37) \\ &= 1 \otimes 1 + i\theta^a (T^a_R \otimes 1 + 1 \otimes T^a_{R'}) && (38) \end{aligned}$$

the last equation can be interpreted as a generalized addition of angular momentum. This can be schematically represented by

$$T^a_{R\otimes R'} = T^a_R + T^a_{R'} \tag{39}$$

Remember that a highest weight state is characterized by

$$E_\alpha hws(R) = 0 \tag{40}$$

it is annihilated by all of the positive simple roots. Consider the state

$$hws(R) \otimes hws(R') \tag{41}$$

It is obviously annihilated by the positive simple roots, so it acts as a highest weight state of the group G on $R \otimes R'$. We can use this highest weight state to build a representation of G. Thus, we can decompose the tensor product into representations of G as follows

$$R \otimes R' \sim \oplus_i R_i \tag{42}$$

where the i are labeled by their highest weight state.

Let us do an example with $SU(N)$, and two copies of the fundamental representation that we tensor together. The highest weight state is $|1\rangle$. Thus, in the tensor product, the highest weight state is

$$|1\rangle \otimes |1\rangle \tag{43}$$

The table of how the lowering operators act on the $|I\rangle$ is simple to write down. We have that

$$\begin{aligned} E_{-1}|1\rangle &= |2\rangle & (44)\\ E_{-2}|2\rangle &= |3\rangle & (45)\\ &\vdots & (46)\\ E_{-(N-1)}|N-1\rangle &= |N\rangle & (47) \end{aligned}$$

all other combinations vanish. This can be simplified to

$$E_{-i}|j\rangle = \delta_{ij}|j+1\rangle \tag{48}$$

The weights of $|1\rangle \otimes |1\rangle$ are the sum of two copies of the weights of $|1\rangle$, and this ends up giving the representation characterized by the obvious highest weight state

$$2 \qquad 0 \qquad 0 \qquad 0 \qquad 0 \tag{49}$$

Now, to build the rest of the representation, we act with the $E_{-\alpha}$. Only E_{-1} can act non-trivially on $|1\rangle$. We find that

$$\begin{aligned} E_{-1}(|1\rangle \otimes |1\rangle) &= (E_{-1} \otimes 1 + 1 \otimes E_{-1})|1\rangle \otimes |1\rangle & (50)\\ &= (E_{-1}|1\rangle) \otimes |1\rangle + |1\rangle \otimes (E_{-1}|1\rangle) & (51)\\ &= |2\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle & (52)\\ &= |1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle & (53) \end{aligned}$$

Now, similarly, we find that

$$E_{-1}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) = |2\rangle \otimes |2\rangle + |2\rangle \otimes |2\rangle = 2|2\rangle \otimes |2\rangle \tag{54}$$

We can now drop the tensor product symbols everywhere to condense notation (it is understood in the obvious way).

So, for example

$$E_{-2}(|1\rangle|2\rangle + |2\rangle|1\rangle) = |1\rangle|3\rangle + |3\rangle|1\rangle \tag{55}$$

And we can continue this way. It is easy to show that we get all of the states of the form

$$|i\rangle|j\rangle + |j\rangle|i\rangle \tag{56}$$

These are all symmetric. There are $N(N+1)/2$ such states, which is not equal to N^2 (the dimension of the tensor product). Thus there must be another representation of lower dimension in the tensor product of two fundamentals.

We notice that in the symmetric tensor only get one state with the weights of $wt(|1\rangle) + wt(|2\rangle)$.

However, in the tensor product there are two such states $|1\rangle|2\rangle$ and $|2\rangle|1\rangle$. We should find the orthogonal state to $|1\rangle|2\rangle + |2\rangle|1\rangle$. We need a linear combination of the form

$$\alpha|1\rangle|2\rangle + \beta|2\rangle|1\rangle \tag{57}$$

requiring orthogonality with the norm we have constructed gives a state where $\alpha = -\beta \simeq= 1$, where we have chosen the normalization of α to be equal to one.

The state we obtain is of the form

$$|1\rangle|2\rangle - |2\rangle|1\rangle \tag{58}$$

and it is antisymmetric in the 12 labels. We can ask if this is a highest weight state. It is easy to check that $E_i|1\rangle|2\rangle - |2\rangle|1\rangle = 0$ for $i > 1$, as it acts by zero on both. The only one that needs checking is E_1. remember that $E_1|2\rangle = |1\rangle$. Thus we get

$$E_1(|1\rangle|2\rangle - |2\rangle|1\rangle) = |1\rangle|1\rangle - |1\rangle|1\rangle = 0 \tag{59}$$

We see that the state above is therefore a highest weight state: it is annihilated by all positive roots. For this highest weight state, we add the weights of $|1\rangle$ and $|2\rangle$ in the Dynkin diagram, and we obtain

$$0 \qquad 1 \qquad 0 \qquad 0 \qquad 0 \tag{60}$$

Now, we can build the descendants by lowering operators. For example

$$E_{-1}(|1\rangle|2\rangle - |2\rangle|1\rangle) = |2\rangle|2\rangle - |2\rangle|2\rangle = 0 \tag{61}$$

and

$$E_{-2}(|1\rangle|2\rangle - |2\rangle|1\rangle) = |1\rangle|3\rangle - |3\rangle|1\rangle \tag{62}$$

In the end, it is also easy to show that we get all state of the form

$$|i\rangle|j\rangle - |j\rangle|i\rangle \tag{63}$$

There are $N(N-1)/2$ such states. If we combine the information from both representations, we find that the tensor product of two fundamentals is a copy of the symmetric plus the antisymmetric combination.

A (horrible) notation would be

$$F \times F = A + S \tag{64}$$

where we use a different label F, A, S for fundamental, symmetric and anti-symmetric. Another standard notation is to denote representations by their dimension. thus, we would have

$$N \times N = \frac{N(N+1)}{2} + \frac{N(N-1)}{2} \tag{65}$$

There is a better notation, which is described by Young Tableaus. Let us introduce a box for each fundamental. The box can take the labels $1, \ldots N$ (or equivalently $|1\rangle \ldots |N\rangle$.

To denote symmetric combinations, we put boxes side by side. To denote antisymmetric combinations, we put boxes on top of one another. Thus, we could have the equation

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \tag{66}$$

These tabelaux can be filled with integers i to indicate vectors. The rules are that the label on the right box has to be greater or equal than the label to it's left, and that the label of a box below another one has to bee strictly greater than the one above.

Thus, for the symmetric representation, we are allowed the labels

$$\begin{array}{cccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} & \ldots & \begin{array}{|c|c|}\hline 1 & \mathrm{N} \\ \hline\end{array} \\
 & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} & \ldots & \begin{array}{|c|c|}\hline 2 & \mathrm{N} \\ \hline\end{array} \\
 & & \ddots & \vdots \\
 & & & \begin{array}{|c|c|}\hline \mathrm{N} & \mathrm{N} \\ \hline\end{array}
\end{array} \tag{67}$$

which gives us the complete collection of states.

Similarly, we are allowed the labels

$$
\begin{array}{cccc}
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} & & & \\
\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} & \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} & & \\
\vdots & \vdots & \ddots & \\
\begin{array}{|c|}\hline 1 \\ \hline N \\ \hline \end{array} & \begin{array}{|c|}\hline 2 \\ \hline N \\ \hline \end{array} & \dots & \begin{array}{|c|}\hline \text{N-1} \\ \hline N \\ \hline \end{array}
\end{array}
\tag{68}
$$

We can now consider the problem of three boxes. We have clearly the highest weight state given by

$$|1\rangle|1\rangle|1\rangle \simeq |111\rangle \tag{69}$$

If we act with E_{-1} lowering operators, again we get states like

$$|112\rangle + |121\rangle + |211\rangle \tag{70}$$

which are completely symmetric in the labels.

To this representation we would associate the Young tableaux

$$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} \tag{71}$$

and the two states would be given by

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} \tag{72}$$

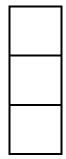
which follow our convention on numbering.

The total number of such states is $N(N+1)(N+2)/6$. It can be related to counting boson states in statistical mechanics, where it becomes a standard formula

The highest weight state has labels

$$
\begin{array}{ccccccccc}
3 & & 0 & & 0 & & 0 & & 0 \\
\bullet & \text{——} & \bullet & \text{——} & \bullet & \text{——} & \bullet & \text{——} & \bullet
\end{array}
\tag{73}
$$

Similarly, we can have a totally antisymmetric representation given by the tableaux



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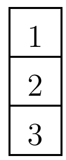
There are a total of $N(N-1)(N-2)/6$ such states. These can be thought of as the number of fermion states. Our rules would require that the labels are strictly increasing downwards, so the minimal labels we can put are



(75)

and in the Dynkin diagrams we would add the weights of states one two and three, giving us

$$0 \quad 0 \quad 1 \quad 0 \quad 0 \tag{76}$$

However, when we apply the theory of descendants, we only get one state with the labels of $|112\rangle$, while there are three such states. We have three such possibilities $|112\rangle, |121\rangle, |211\rangle$.

We should again pick an orthogonal state to $|112\rangle + |121\rangle + |211\rangle$. Now, we find that there are two orthogonal states with those same weights. So we should have two copies of the corresponding representation.

The best choice is to pick the simplest combination that does the trick. We begin with the ordering given by

$$|11; 2\rangle \tag{77}$$

and we take an antisymmetric combination of the form

$$|11; 2\rangle - |21; 1\rangle \tag{78}$$

where we antisymmetrize in the first label after the semicolon. This is orthogonal to $E_{-1}|111\rangle$. Notice that the first state we wrote is symmetric on the first two indices, but we antisymmetrized later and lost this property. This is a general feature. To this object we associate the tableaux

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \tag{79}$$

This is symmetric in the elements of the first row, and antisymmetric in the elements of the first column.

The highest weight state is described by the labeling

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \tag{80}$$

where we use the rules requiring that numbering on a column is strictly increasing. To this object we associated the state

$$|11;2\rangle - |21;1\rangle \tag{81}$$

where we antisymmetrize over the corresponding columns. The semicolon tells us where to make the breaks to match the Young tableaux in an obvious way. The corresponding representation would be characterized by the highest weight state

$$1 \qquad 1 \qquad 0 \qquad 0 \qquad 0 \tag{82}$$

For example, the following labels are allowed

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \tag{83}$$

but the labels

$$\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \tag{84}$$

are not. This labeling would give rise to the same state as before and to a net overcounting of states.

For the first one, we would get

$$|14;2\rangle + |41;2\rangle \tag{85}$$

after symmetrization in the first row. This is invariant under the exchange of $1,4$. Then we need to anstisymmetrize on columns to get that the tableaux with those labels would correspond to the state

$$|14;2\rangle - |24;1\rangle + |41;2\rangle - |21;4\rangle \tag{86}$$

Notice that after antisymmetrization the object is not symmetric in the first two indices any longer, but it is symmetric in the labels $1,4$.

You should also check that this object is orthogonal to

$$\begin{ytableau} 1 & 2 & 4 \end{ytableau} \tag{87}$$

which is built by symmetrizing in 124 (this produces 6 permutations).

As the set of weights of the form 112 has another state, there is a second representation with the same weight as the highest weight we found. Thus there are two copies of the same representation associated to a corner. This saturates the set of representations.

Thus we can have the equation

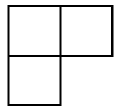
$$\dim\left(\ydiagram{1}\right)^3 = \dim\left(\ydiagram{3}\right) + \dim\left(\ydiagram{1,1,1}\right) + 2\dim\left(\ydiagram{2,1}\right) \tag{88}$$

Giving us the result that

$$\dim\left(\ydiagram{2,1}\right) = \frac{N(N-1)(N+1)}{3} \tag{89}$$

2 Hooks and general highest weight states

Consider again the tableaux given by



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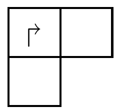
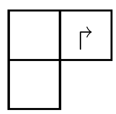
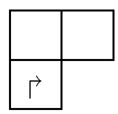
For each box we can define a hook



(91)

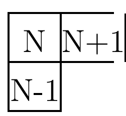
which is a path that goes upward on the diagram and turns right at he box. The length of a hook is the number of boxes that it crosses. In the example above the length of the hook is three. For the other hooks we get

 , 

(92)

that their hook lengths are one.

Now, fill the boxes of the tableaux with the following labels



(93)

starting with N in the left upper corner, and adding one as we go to the right, while subtracting one ass we go down. These labels are not for states.

The dimension of a representation associated to a Young tableaux R can be calculated by the following rule

$$\dim(R) = \prod_{boxes} \frac{N_b}{h_b} \tag{94}$$

where N_b are the labels $N, N = 1$, etc, and h_b is the length of the hook.

All representations of $SU(N)$ can be written in this form. One quickly finds out that the representation of the symmetric and antisymmetric objects with three boxes comes out right, as well as the one above (which we have calculated already). For example, for

$$\dim \begin{array}{|c|c|c|}\hline N & N+1 & N+2 \\ \hline N-1 & N \\ \cline{1-2} N-2 & N-1 \\ \cline{1-2}\end{array} = \frac{N(N+1)(N+2)(N-1)N(N-2)(N-1)}{5\times 4\times 1\times 3\times 2\times 2\times 1} \tag{95}$$

For a general tableaux, the highest weight state is obtained by filling the first row with ones, the second row with twos, the third row with threes, etc. This is the minimal filling consistent with our rules. Thus, for the example above we get

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2}\end{array} \tag{96}$$

Thus, we would begin with the state given by

$$|111;22;33\rangle \tag{97}$$

To build the highest weight state we are instructed to antisymmetrize on columns. These are permutations of the labels in the columns with the sign of the permutation included. Thus, we have that

$$hws(3|3|1) = \sum_{\sigma\in\pi(Col)} (-1)^\sigma \sigma|111;22;33\rangle \tag{98}$$

where σ acts on the Hilbert space by permutation of the vectors in the columns. This action commutes with the action of the Lie algebra on the set of states.

For example, antisymmetrizing on the first column would lead to a state of the form

$$|111;22;33\rangle - |211;12;33\rangle - |311;22;13\rangle - |111;32;23\rangle + |211;32;13\rangle + |311;12;13\rangle \tag{99}$$

This state would be further antisymmetrized on the second column, turning each of the kets above into 6 different vectors (giving a total of 36 kets contributing to the sum). It is a lot easier to present the state as in equation 96, which contains all of these sums implicitly.

3 Tensor products of two tableaux

Let us consider now tensor products of representations. For example, let us consider a tensor product of two symmetric tensors of $SU(N)$. The hws of the symmetric representation is $|11\rangle$, so the highest weigh state of the tensor product will be given by

$$|11\rangle \otimes |11\rangle \sim |1111\rangle \tag{100}$$

Thus we find that in the tensor product there is a four-tensor with symmetric indices. We can also consider states of the form

$$|11\rangle \otimes |22\rangle \sim |11;22\rangle \tag{101}$$

which are obviously symmetric in the first pair of two indices and the second pair of indices. However, we should consider states with only one two and three ones. There are two such states, given by

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \tag{102}$$

But only one of them is a descendant of $|11\rangle \otimes |11\rangle$. The orthogonal combination can be shown to be a highest weight state. Thus, if we consider the possibilities above we find that

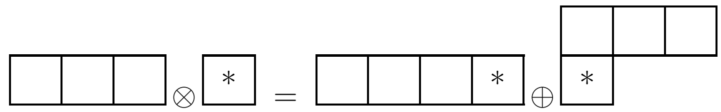
$$\begin{array}{|c|c|}\hline & \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline * & * \\ \hline\end{array} \sim \begin{array}{|c|c|c|c|}\hline & & * & * \\ \hline\end{array} \tag{103}$$

$$\oplus \begin{array}{|c|c|}\hline & \\ \hline * & * \\ \hline\end{array} \tag{104}$$

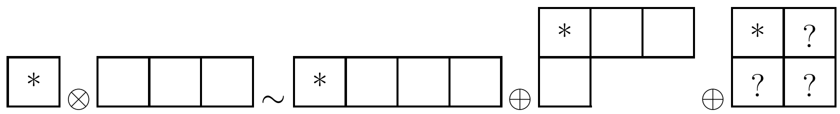
$$\oplus \begin{array}{|c|c|c|}\hline & & * \\ \hline * \\ \cline{1-1}\end{array} \tag{105}$$

where we have indicated how we stack the boxes from the second tableaux with asterisks. We see that objects in the same row are allowed to be in different rows in the tensor product. This is, we are allowed to cross-antisymmetrize between elements of the first tableaus and the second one.

The tensor product is commutative (there is an obvious way to relate one basis to the other and to show that the Lie algebra is the same). Thus, consider for example


(106)

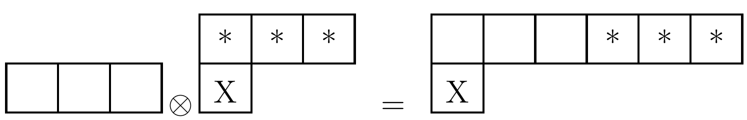
If we do it in the opposite order, we find that it would be in principle possible to write three tabelaux

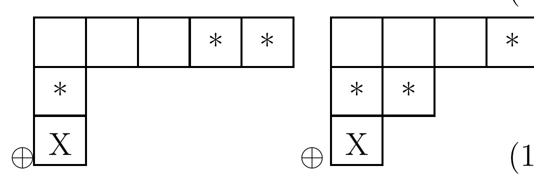

(107)

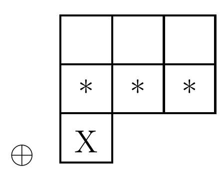
by allowing the stacking. But this third one can not be allowed if the tensor product is commutative. We see the reason why is obvious on closer inspection. In the left hand side the unmarked boxes are all symmetric, but in the third diagram we are trying to antisymmetrize them further. Such a process gives zero. Thus we find that we are not allowed to move a box that started in the same row as another box, to a box that is in the same column after the tensor product.

There is a similar rule for boxes that begin in the same column: you are not allowed to move them so that they appear on the same row as another such box. These rules saturate the possibilities, while taking into account the minimal ordering of labels that we can have.

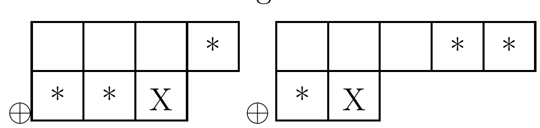
For example, consider the products


(108)


(109)


(110)

But we then realize that the following are also allowed


(111)

This can be verified easily for $SU(2)$ representations, where objects antisymmetric in three indices are identically zero, and where objects antisymmetric in two indices are trivial. This is because the box marked with X never shows up in the same row as the first box that is marked with an asterisk.

You can verify that the dimensions of the representations add up to each other.

from the point of view of HWS, we find that the labels

$$|111\rangle \otimes |122; 2\rangle, |111\rangle \otimes |112; 2\rangle \tag{112}$$

show up in various places and can be symmetrized accordingly.

Again, you should consult books on group theory for more information.

4 Branching rules

We want to consider standard splitting of a group reprsentation into representations of subgroups $SU(N) \to SU(M) \times SU(N-M) \times U(1)$ into a block diagonal form, where we have a standard embedding. This is schematically

$$\begin{pmatrix} SU(M) & 0 \\ 0 & SU(N-M) \end{pmatrix} \tag{113}$$

where the extra $U(1)$ is embedded as

$$\begin{pmatrix} \text{diag}(1/M) & 0 \\ 0 & \text{diag}(-1/(M-N)) \end{pmatrix} \tag{114}$$

It is easy to realize how to embed the Dynkinm diagrams of $SU(M)$ and $SU(N_M)$ into the one of $SU(N)$. After all, we have an obvious identification of the roots and the Cartan. We find that the split is obtained by deleting one node (the one connecting the $SU(M)$ and the $SU(N-M)$). Schematically, this is

$$\bullet - \bullet - \bullet - \otimes - \bullet - \bullet - \bullet \tag{115}$$

where the node demarcated with the $\otimes$ is to be deleted. Above we have the splitting $SU(8) \to SU(4) \times SU(4) \times U(1)$. How to treat the $U(1)$ is hidden. Let us consider splitting the fundamental representation. It should be clear that

$$\square_N \sim \square_M \oplus \square_{N-M} \tag{116}$$

We can follow the weights of the different states to to this split.

$$\begin{array}{ccccccccccccc} 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\ \bullet & - & \bullet & - & \bullet & - & \otimes & - & \bullet & - & \bullet & - & \bullet \end{array} \tag{117}$$

This is the highest weight state of the fundamental of $SU(M)$. As we consider the weights of other elements we eventually find the state

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 0 & & -1 & & 1 & & 0 & & 0 \\ \bullet & - & \bullet & - & \bullet & - & \otimes & - & \bullet & - & \bullet & - & \bullet \end{array} \tag{118}$$

where the -1 is over the deleted node and therefore does not count. We see that this is a highest weight state with respect to the $SU(N-M)$.

We can do this similarly for the antifundamental. (Just change signs for all weights). The highest weight state of the complex conjugate representation comes from flipping the diagram on the horizontal direction with the highest weight states labels.

For example, consider the highest weight state of the adjoint representation

$$\begin{array}{ccccccccccccc} 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 \\ \bullet & - & \bullet & - & \bullet & - & \otimes & - & \bullet & - & \bullet & - & \bullet \end{array} \tag{119}$$

This is a hws for a product

$$\square_M \otimes \overline{\square}_{N-M} \tag{120}$$

One also finds the following hws in the product

$$\begin{array}{ccccccccccccc} 1 & & 0 & & 1 & & -1 & & 0 & & 0 & & 0 \\ \bullet & - & \bullet & - & \bullet & - & \otimes & - & \bullet & - & \bullet & - & \bullet \end{array} \tag{121}$$

and

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 0 & & -1 & & 1 & & 0 & & 1 \\ \bullet & - & \bullet & - & \bullet & - & \otimes & - & \bullet & - & \bullet & - & \bullet \end{array} \tag{122}$$

These two are the adjoint of $SU(M)$ and the adjoint of $SU(M-N)$. We see that whenever we start moving the weights to the deleted node we get transitions between different representations. We also have the hws state

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 1 & & -2 & & 1 & & 0 & & 0 \\ \bullet & — & \bullet & — & \bullet & — & \otimes & — & \bullet & — & \bullet & — & \bullet \end{array} \tag{123}$$

which corresponds to the complex conjugate of the first representation.

There is an extra singlet that appears in this product (this can be seen by adding the dimensions of the representations and finding one missing element.). This can be also understood from

$$Adj_{SU(N)} \simeq N \otimes \bar{N} - \text{trace} \tag{124}$$

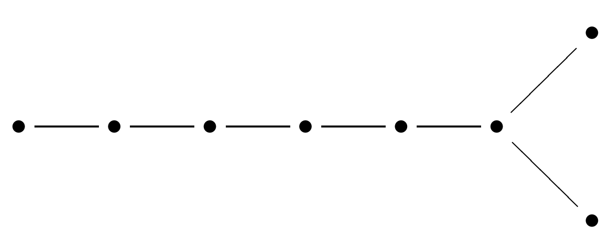
So that when we split, we find that

$$Adj_{SU(N)} \simeq (M \oplus (N-M)) \otimes (\bar{M} \oplus \overline{N-M}) - \text{trace} \tag{125}$$

IN the product we get two adjoints, and each of them involves removing a trace. However, we have subtracted only one trace in $Adj_{SU(N)}$, while we have subtracted two traces in $Adj_{SU(M}$ and $Adj_{SU(N-M)}$. Thus, the left hand side must have an extra trace. This is just the $U(1)$ we had above. The off-diagonal elements of the matrix decomposition are those that are charged under both gauge groups.

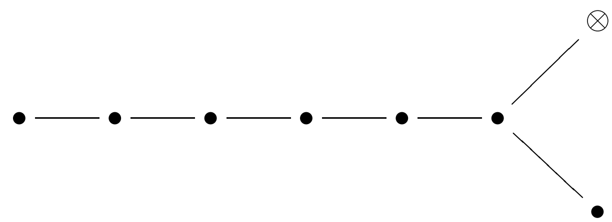
5 Some other useful Dynkin diagrams and hws.

The Dynkin diagram for $SO(2N)$ is represented by



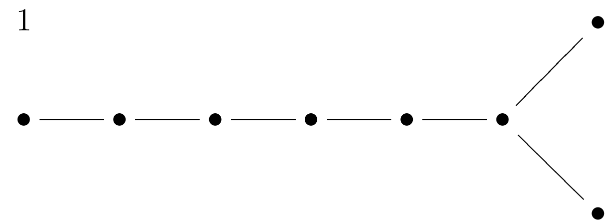
(126)

There is an obvious $SU(N)$ subgroup by deleting a node



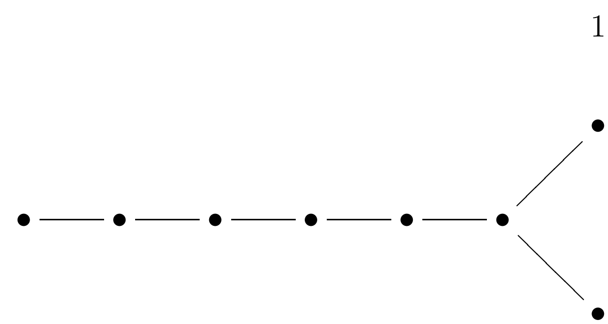
(127)

The HWS of the fundamental is given by



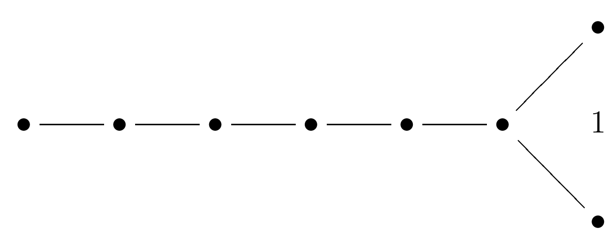
(128)

While for the spinor it is given by



(129)

The complex conjugate spinor has HWS given by

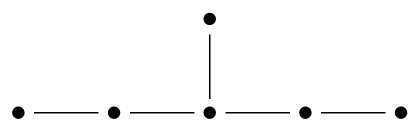


(130)

The complex conjugation symmetry is obtained by reflection along a horizontal line (an obvious symmetry of the diagram). The fact that the HWS of the fundamental is invariant implies that the fundamental representation is real.

Under the $SO(2N) \to SU(N)$ splitting, the fundamental splits into a fundamental plus an antifundamental of $SU(N)$. This lets one fill in all of the weights in a unique way.

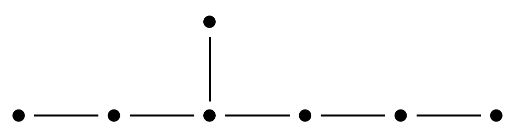
The Dynkin diagram for E_6 is given by



(131)

It has an obvious $SU(6)$ subgroup and an obvious $SO(10)$ subgroup. It also has an obvious reflection, which corresponds to complex conjugation.

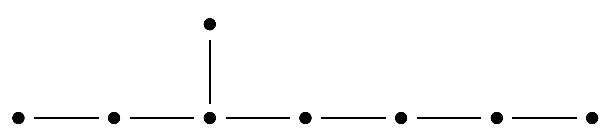
The Dynkin diagram for E_7 is similar, given by



(132)

It has an obvious E_6 and an obvious $SU(7)$ symmetry. It also has an obvious $SO(12)$ subgroup. There is no reflection symmetry associated to complex conjugation, so all of the representations are necessarily pseudo-real.

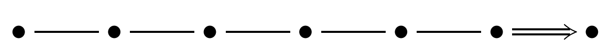
Finally, the Dynkin diagram for E_8 is given by



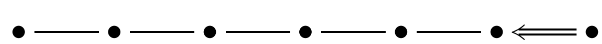
(133)

The diagram has an obvious E_7 subgroup symmetry, an obvious $SU(8)$, and an obvious $SO(14)$, depending on which node one deletes first. One also has an obvious split into $E_6 \times SU(2)$.

Finally, there are non-simply laced groups $SO(2N+1)$ and $Sp(N)$ (the so called B,C groups). Each of these is of rank N (the number of nodes) They have long roots and short roots. These are indicated by arrows



(134)



(135)

Notice also the obvious identities of diagrams $Sp(2) \sim SO(5)$, $SU(4) \sim SO(6)$, $SO(3) \sim SU(2) \sim Sp(1)$. These are the classical coincidences of various series of groups. Under this rubric the exceptional series would have $E_5 \sim SO(10)$, $E_4 \sim SU(3) \times SU(2)$, $E_3 \sim SU(2) \times SU(2)$ (these sometimes show with this name in string theory)